SEC. 1.2

8. \( e^h = 1 + h + h^2/2! + \cdots + 1 + O(h^4) = 1 + o(1). \) 
   (1 - h^4)^{-1} = 1 + h^4 + h^8 + \cdots = 1 + O(h^4) = 1 + o(h^3). 
   \cos(h) = 1 - h^2/2! + h^4/4! + \cdots = 1 + O(h^4) = 1 + o(h^2). 
   1 + \sin(h^2) = 1 + h^2 - h^4/3! + \cdots = 1 + O(h^4) = 1 + o(h^2). 

9. \( e^h = 1 + h + h^2/2! + h^3/3! + \cdots \) so \( [(1 + h) - e^h]/h^2 = -1/2 - h/6 - \cdots \) as \( h \to 0. \) Hence, \( -1/2 + O(h^2) \) with \( \beta = 1 \) the best value and \( -1/2 + o(h^2) \) with \( \beta = 0 \) the best value.

22. Put \( x_n = (1 + a \theta^n)/(1 + a \theta^{n-1}). \) We want to find \( c \) and \( N \) so that \( 0 < c < 1 \) and \( n \geq N \) implies \( |x_{n+1} - 1| < c|x_n - 1| \). We have
   \[
   \frac{x_{n+1} - 1}{x_n - 1} = \frac{[(1 + a \theta^{n+1})/(1 + a \theta^n)] - 1}{[(1 + a \theta^n)/(1 + a \theta^{n-1})] - 1} = \frac{(1 + a \theta^{n+1} - 1 - a \theta^n)/(1 + a \theta^n)}{(1 + a \theta^n - 1 - a \theta^{n-1})/(1 + a \theta^{n-1})} = \frac{a \theta^n (\theta - 1)/(1 + a \theta^n)}{a \theta^{n-1} (\theta - 1)/(1 + a \theta^{n-1})} = \frac{\theta}{1 + a \theta^n}
   \]

The term \( \theta[(1 + a \theta^{n-1})/(1 + a \theta^n)] \) converges to \( \theta. \) Select \( N \) so that for \( n \geq N, \) we have \( \theta[(1 + a \theta^{n-1})/(1 + a \theta^n)] < \frac{1}{2}(\theta + 1). \) Then we can use \( c = \frac{1}{2}(\theta + 1) < 1 \) for \( n \geq N. \)

SEC. 1.3

10. The null space of \( E^r \) consists of sequences \( x = [x_1, x_2, \ldots] \) in which \( x_i = 0 \) for \( i > r. \) The null space has dimension \( r. \)

11. a. Characteristic equation: \( x^3 - 3x^2 + 4 = 0. \) Roots: \(-1, 2 \) (double).
    Basis: \([-1, 1, -1, \ldots, (-1)^n, \ldots, [2, 4, 8, 16, \ldots, 2^n, \ldots], [1, 4, 12, 32, \ldots, n2^{n-1}, \ldots].
    b. Characteristic equation: \( 3x^2 - 2x + 3 = 0. \) Roots: \(1 \pm i \sqrt{2}. \)
    Basis: \( u_n = (1 + i \sqrt{2})^n, v_n = (1 - i \sqrt{2})^n. \)
    c. Characteristic equation: \( 2x^6 - 9x^5 + 12x^4 - 4x^3 = 0. \) Roots: \(0 \) (triple), \(1/2 \) (simple), \(2 \) (double).
    Basis: \( x^{(1)} = [1, 0, 0, \ldots], x^{(2)} = [0, 1, 0, \ldots], x^{(3)} = [0, 0, 1, 0, \ldots], x^{(4)} = [1, 1/2, 1, 1, 1/2, 1, \ldots], x^{(5)} = [2, 4, 8, \ldots], x^{(6)} = [1, 4, 12, \ldots]. \) Here the general term for \( x^{(6)} \) is \( x^{(6)}_n = n2^{n-1}. \)

13. Notice these are not polynomial difference operators. Solve by inspection for first few terms.
   a. \( x_{n+1} = n!x_1 \)    b. \( x_{n+1} = (1/2)n(n+1) + x_1 \)    c. \( x_{n+1} = 2n + x_1 \)

14. It is obvious that \( \Delta = E - I. \) If \( p \) is a polynomial of degree \( n, \) then by Taylor’s Theorem \( p(x) = \sum_{j=0}^{n} p^{(j)}(a)/j! (x-a)^j. \) Put \( x = E \) and \( a = I \) to get \( p(E) = \sum_{j=0}^{n} (1/j!)p^{(j)}(I) \Delta^j. \)

27. Characteristic equation: \( \lambda^2 - 2\lambda - 2 = 0. \) Roots: \(1 \pm \sqrt{3}. \)
    General solution: \( x_n = \alpha(1 + \sqrt{3})^n + \beta(1 - \sqrt{3})^n. \)
    Initial values give \( 1 = x_1 = \alpha(1 + \sqrt{3}) + \beta(1 - \sqrt{3}) \) and \( 1 - \sqrt{3} = x_2 = \alpha(1 + \sqrt{3})^2 + \beta(1 - \sqrt{3})^2. \)
    So solution is \( \alpha = 0 \) and \( \beta = 1/(1 - \sqrt{3}). \)