6.3 Hermite Interpolation

1. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-9</td>
</tr>
<tr>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>44</td>
</tr>
</tbody>
</table>

So $p(x) = -2 + 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$.

3. By Theorem 1, there exists a unique polynomial $p$ of degree $\leq m (m = 2n + 1)$ such that $p(x_i) = y_i$ and $p'(x_i) = 0$ for $0 \leq i \leq n$. By Equation (9), we have $p(x) = \sum_{i=0}^{n} y_i [1 - 2(x-x_i)\ell_i'(x_i)]\ell_i^2(x)$ where $\ell_i(x) = \prod_{j=0, j\neq i}^{n} (x-x_j)/(x_i-x_j)$ for $0 \leq i \leq n$.

4. Let us write $p(x) = a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3$. Then $p'(x) = 2c + 6d(x-x_0)$. The four conditions can be written as: $c_{00} = p(x_0) = a, c_{02} = p''(x_0) = 2c, c_{10} = p(x_1) = a + bh + ch^2 + dh^3$, and $c_{12} = p''(x_1) = 2c + 6dh$ when $h = x_1 - x_0$. So $a$ and $c$ are obtained without restrictions: $a = c_{00}, c = c_{02}/2$. $d$ and $b$ can be obtained from last two equations: $\begin{bmatrix} h & h^3 \\ 0 & 6h \end{bmatrix} = \text{known vector}$.

\[
\text{Det} \begin{bmatrix} h & h^3 \\ 0 & 6h \end{bmatrix} = 6h^2 \neq 0 \text{ if } h \neq 0 \Rightarrow \text{condition: } x_0 \neq x_1.
\]

6.4 Spline Interpolation

5. $f(1^-) = 1 = f(1^+)$, so $f$ is continuous at $x = 1$. Also $f(2^-) = 3/2 = f(2^+)$, so $f$ is continuous at $x = 2$. $f'(x) = \begin{cases} 1 & x \in (-\infty, 1] \\ 2 - x & x \in [1, 2] \\ 0 & x \in [2, \infty) \end{cases}$.

Thus, $f'(1^-) = 1 = f'(1^+)$ and $f'(2^-) = 0 = f'(2^+)$. Therefore, $f'(x)$ is continuous at $x = 1, x = 2$.

Hence, $f$ is a quadratic spline function.

7. Enforce the continuity of $f$ at knots: $1, 3$. At $x = 1, a(-1)^2 + 0 = c(-1)^2 \Rightarrow a = c$. At $x = 3, c(3)^2 = d(1)^2 + 0 \Rightarrow c = d$. Continuity of $f'$ at knots: At $x = 1, 2a(-1) + 0 = 2c(-1) \Rightarrow a = c$. At $x = 3, 2c = 2d + 0 \Rightarrow c = d$. Continuity of $f''$ at knots: At $x = 1, 2a + 0 = 2c \Rightarrow a = c$. At $x = 3, 2c = 2d + 0 \Rightarrow c = d$. Thus, in order that $f$ be a cubic spline, $a = c = d$ and $b, e$ any arbitrary values. Next, determine $a, b, c, d, e$ so that $f$ interpolates the table. At $x = 0, a(-2)^2 + b(-1)^3 = 26 \Rightarrow 4a - b = 26$. At $x = 1, a(-1)^2 + b \cdot 0 = 7 \Rightarrow a = 7$. So $b = 2$ and $c = d = 7$.

At $x = 4, d(2)^2 + e(1)^3 = 25 \Rightarrow 28 + e = 25 \Rightarrow e = -3$. Then: $a = c = d = 7, b = 2, e = -3$.

9. Put $q_i(x) = \frac{1}{2}(z_i + h_i)(x - ti)^2 - \frac{1}{2}(z_i - h_i)(t_i + 1 - x)^2 + y_i + \frac{1}{2}z_i h_i$.

Then $q_i(t_i) = \frac{1}{2}(z_i + h_i)(t_i^2 - 1) + y_i + \frac{1}{2}z_i h_i = \frac{1}{2}[-(z_i h_i^2)/h_i] + \frac{1}{2}(z_i h_i) + y_i = y_i$, where $h_i = t_{i+1} - t_i$, $q_i'(x) = (z_i + h_i)(x - t_i) + (z_i - h_i)(t_i - x)$, $q_i''(t_i) = (z_i + h_i)(t_i - x) = z_i$, $q_i'(t_i) = (z_i + h_i)(t_i + 1 - x) = z_i$, $q_i''(t_i) = (z_i + h_i + y_i) + (z_i - h_i + y_i)$, $q_i'(t_i) = (z_i + h_i + y_i) + (z_i - h_i + y_i)$, $q_i''(t_i) = (z_i + h_i + y_i) + (z_i - h_i + y_i)$.

Continuity Condition $1/2(z_i + 1)h_i - 1 + y_i = y_i \Rightarrow z_i + 1 = -h_i - h_i(1)(y_i - 1)$. $q_i(x) = \frac{1}{2}(z_i + h_i)(x - t_i - h_i)^2 - \frac{1}{2}(z_i - h_i)(t_i - x - h_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}(z_i + h_i)(x - t_i - h_i)^2 - \frac{1}{2}(z_i - h_i)(t_i - x - h_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}(z_i + 1)h_i + y_i$.
Here $i = 1, 2, \ldots, n - 1$. $Q$: piecewise quadratic $Q_i$, $Q'$ continuous $Q_i'(t_i) = z_i$ well-defined
$q_i(t_2) = q_2(t_2)$ etc. $q_{n-2}(t_{n-1}) = q_{n-1}(t_{n-1})$, i.e.: $q_{n-1}(t_i) = q_i(t_i)$ for $i = 2, \ldots, n - 1$ $z_{i-1} + z_i = (2/h_{i-1})(y_i - y_{i-1})$ ($2 \leq i \leq n - 1$).
Let $z_1 = 0$ and define inductively $z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$, $i = 2, 3, \ldots, n$. $z_i$ is arbitrary,
$z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$, $i = 2, \ldots, n$.
So $z_i = a_i - z_{i-1}$. $z_2 = a_2 - z_1$. $z_3 = a_3 - z_2 = a_3 - (a_2 - z_1)$ = $a_3 - a_2 + z_1$. $z_4 = a_4 - z_3 = a_4 - a_3 + a_2 + z_1$, Etc.$\cdots z_i = a_i - z_{i-1} - a_{i-2} \cdots + (-1)^i(a_2 - z_1)$.
So $z_i = \gamma_i - (-1)^i z_1$, $\gamma_2 = a_2$, $\gamma_3 = a_3 - a_2$, $\gamma_4 = a_3 - a_2 + a_1$, $\gamma_n = a_3 - a_2 + \cdots + (-1)^i(a_2 - z_1)$.

$(n-1)z_1 - (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n) = 0$ $z_1 = (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n)/(n-1)$.

Now $\gamma_2 - \gamma_3 = a_2 - (a_3 - a_2) = 2a_2 - a_3$ and $\gamma_4 - \gamma_5 = \gamma_4 - (a_4 - a_2) = 2\gamma_4 - a_2$. $\gamma_2 = a_2$, $\gamma_3 = a_3 - a_2$, $\gamma_4 = a_4 - a_3 + a_2$, $\gamma_5 = a_5 - a_4 + a_3 - a_2$, Etc. $\gamma_2 = a_2$, $\gamma_3 = a_3 - a_2$, $\gamma_4 = a_4 - a_3 + a_2$, $\gamma_5 = a_5 - a_4 + a_3 - a_2$, Etc. So $[\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n]/(n-1)$ = $[(n-1)z_1 - (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n)/(n-1)$.

Algorithm: For $i = 2 \cdots n$ define $a_i = (2/h_{i-1})(y_i - y_{i-1})$. For $i = 3 \cdots n$ do $a_i - a_{i-1} \rightarrow a_i$. For $i = 2 \cdots n$ do $a_0 - (-1)^i a_{i+1} \rightarrow a_i$. $z_1 = a_2/(n-1)$.

13. Let $f_1(x) = 1 + x - x^3$, $x \in [0, 1]$; $f_2(x) = 1 - 2(x - 1) - 3(x - 1)^2 + 4(x - 1)^3$, $x \in [1, 2]$;
$f_3(x) = 4(x-2)+6(x-2)^2-3(x-2)^3, x \in [2, 3]$. Since $f_1(0) = 1$, $f_1(1) = 1 = f_2(1)$, $f_2(2) = 0 = f_3(2)$,
and $f_3(3) = 10$, $f(x)$ interpolates the table and $f(x)$ is continuous at $x = 1$ and $x = 2$. Also
$f_1'(1) = -2 = f_2'(1), f_2'(2) = 4 = f_3'(2), f_3'(3) = -6 = f_4'(1), f_4'(2) = 18 = f_5'(2), f_5'(3) = 0, 0,$
and $f_5'(3) = 0$. Hence, $f(x)$ is a natural cubic spline which interpolates the table values.

24. The integral $\int_0^1 S(x) dx$ becomes an approximation of the area under the curve $f(t)$ on the interval
$[0, 1]$ where $R_i = \int_0^1 [f(t_i) + f(t_{i-1})]$. This is the composite trapezoid rule with non-uniform
segments. Thus, $\int_0^1 S(x) dx = \sum_{i=1}^n R_i = \frac{1}{2} \sum_{i=1}^n (t_i - t_{i-1})[f(t_i) - f(t_{i-1})]$.

30. In the matrix system for the $z_i$'s in the text, the first equation is replaced by $h_{2i} + u_{2i}1 + h_{2i+1} = v_1$
and the last equation is replaced by $h_{n-1}z_{n-1} + u_{n-1}z_{n-1} + h_{n-2}z_n = v_{n-1}$. Using Eq. (8) with
$i = 0$, we have $-h_{2i} + u_{2i}1 + h_{2i+1} = 6S(t_0) - 60$. Using Eq. (9) with $i = n$, we have $h_{n-1}z_{n-1} + 2h_{n-2}z_n = 6S(t_n) - b_n$. Use these latter two equations to expand the linear system and solve for
$[z_0, z_1, \ldots, z_{n-2}, z_{n-1}, z_n]^T$.

6.5 The B-Splines: Basic Theory

3. $S(t_m) = \sum_i c_i B_i^2(t_m) = c_{n-2}B_{n-2}^2(t_m) + c_{n-1}B_{n-1}^2(t_m)$
$= c_{n-2}h_m/(h_m + h_{m-1}) + c_{n-1}h_{m-1}/(h_m + h_{m-1}) = (c_{n-2}h_m + c_{n-1}h_{m-1})/(h_m + h_{m-1}) = y_m$. 

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