1. (5 points) How would you compute \( f(x) = \sqrt{1+x} - \sqrt{1-x} \) when \( x \) is small? why?

\[
f(x) = \sqrt{1+x} - \sqrt{1-x} = \frac{(1+x) - (1-x)}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2x}{\sqrt{1+x} + \sqrt{1-x}}
\]

To avoid subtracting nearly equal #s: \( \sqrt{1+x}, \sqrt{1-x} \) when \( x \) is small.
2. (5 points) Give an argument on why the Newton’s method converges quadratically by interpreting it as a fixed point iteration.

Newton’s method: \[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Let \[ F(x) = x - \frac{f(x)}{f'(x)} \]

then Newton’s method is a fixed iteration:

\[ x_{n+1} = F(x_n) \]

Since \[ F'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \]

If \( f'(x) \to 0 \) and \( f'(x) \neq 0 \) \( \Rightarrow F'(x) = 0 \)

\[ F''(x) = \frac{(f''(x) + f f''(x)) f'(x) - f f' f'' f'}{(f'(x))^4} \]

\( \Rightarrow F''(x) = \frac{f''(x)}{f'(x)} \neq 0 \) if \( f'(x) \neq 0 \)

So Newton’s method converges at least quadratically.
3. (10 points) Show that the sequence \( x_n \) defined by

\[
x_{n+1} = \frac{2x_n + 3}{x_n + 2}
\]

converges for any \( x_0 \in [-1, \infty) \), and determine its convergence rate. What is \( \lim_{n \to \infty} x_n \)?

If \( x_n \to r \) \( \Rightarrow r = \frac{2r + 3}{r + 2} \)

\( \Rightarrow r = -1 \pm \sqrt{3} \) but since \( x_0 \in [-1, \infty) \)

\[
\lim_{n \to \infty} x_n = \sqrt{3}
\]

Let \( f(x) = \frac{2x + 3}{x + 2} \) \( \Rightarrow f'(x) = \frac{1}{(x+2)^2} \)

Hence \( f'(x) > 0 \) for \( x \in (-1, \infty) \). For \( x_0 \in [0, +\infty) \)

we have \( x_n \geq 0 \),

so \( f(x) \) is a contractive mapping from \([0, +\infty)\)

to \([0, +\infty)\)

and has a fixed point.

Since \( f'(\sqrt{3}) = 0 \), the convergence rate

is linear.
4. (10 points) Find the polynomial $p_4$ (of degree less or equal than 4) in Newton’s form using the divided difference table such that

$$p_4(0) = 1, \quad p_4(1) = 0, \quad p_4'(1) = 2, \quad p_4''(1) = 2, \quad p_4(2) = 3.$$ 

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Hence

$$p_4(x) = 1 - x + 3x(x-1) - 2x(x-1)^2 + x(x-1)^3$$
5. (10 points) Let $p_n(f; x)$ be the polynomial of degree $\leq n$ interpolating $f(x) = e^{2x}$ at $x_i = \cos \frac{2i+1}{2n+2} \pi$, $i = 0, 1, \ldots, n$. Derive an upper bound for $\|f - p_n\|_{L^\infty(-1,1)}$, and determine the smallest $n$ such that the above error is less than $\frac{1}{2} \times 10^{-2}$.

\[
\frac{f(x) - p_n(x)}{(n+1)!} = \frac{f^{(n+1)}(x)}{(n+1)!} \sum_{i=0}^{n} \frac{1}{(n+1)!} (x-x_i)
\]

\[
f^{(n+1)}(x) = 2^{n+1} e^{2x} \Rightarrow \max_{x \in [-1,1]} |f^{(n+1)}(x)| \leq e^2 2^{n+1}
\]

\[
\max_{x \in [-1,1]} \left| \frac{1}{(n+1)!} (x-x_i) \right| \leq \frac{1}{2^n} \quad \text{(Chebyshev points)}
\]

\[
\Rightarrow \left| f(x) - p_n(x) \right| \leq \frac{2 e^2}{(n+1)!} \leq \frac{1}{2} \times 10^{-2}
\]

we need $(n+1)! \geq 400 e^2$

the smallest is $n = 6$. 

5
6. (10 points) Given $x_0 < x_1 < \cdots < x_n$. Determine a procedure for constructing a quadratic spline $S(x)$ such that

$$S(x_i) = f(x_i), \quad i = 0, 1, \cdots, n; \quad S'(x_n) = 0.$$

Let $z_i = S'(x_i)$, $h_i = x_{i+1} - x_i$.

\[ S_i'(x) = \frac{z_i}{h_i} (x_{i+1} - x) + \frac{z_{i+1}}{h_i} (x - x_i) \]

Integrate once \( \Rightarrow \)

\[ S_i(x) = -\frac{z_i}{2h_i} (x_{i+1} - x)^2 + \frac{z_{i+1}}{2h_i} (x - x_i)^2 + C_i \]

From $S_i(x_i) = f(x_i) \Rightarrow C_i = y_i + \frac{h_i}{2} z_i$.

From $S_i(x_{i+1}) = f(x_{i+1}) \Rightarrow y_{i+1} = \frac{z_{i+1}}{2} h_i + C_i = \frac{2i+1}{2} \left( h_i + y_i \right), \quad i = 0, 1, \cdots, n-1$

**Algorithm:**

\[ z_n = 0 \]

\[ \text{for } i = n-1, n-2, \ldots, 0 \quad \text{do} \]

\[ z_i = -2z_{i+1} + \frac{2}{h_i} (f(x_{i+1}) - f(x_i)) \]

\[ \text{end do} \]