1. Write down a detailed proof for Theorem Problem 7.8 on Page 265.

2. Consider the Allen-Cahn equation:

\[ u_t - \Delta u + \frac{1}{\varepsilon^2} (u^3 - u) = 0, \quad (x, y) \in (-1, 1)^2; \quad \frac{\partial u}{\partial n}|_{\partial \Omega} = 0, \]

with the initial condition \( u(x, y, 0) = u_0(x, y) \).

(i) Write down a scheme which is first-order semi-implicit in time and your choice of spectral method in space (e.g., you can use the Legendre-Galerkin solver in Proj 4); and write a program for the scheme.

(ii) Take \( u_0(x, y) = \tanh \left( \sqrt{\frac{x^2 + y^2}{\varepsilon}} - 0.5^2 \right) \) with \( \varepsilon = 0.04 \). Use 65 × 65 points and \( \Delta t \) sufficiently small for the scheme to be stable. Plot the levelset \( u(x, y, t) = 0 \) at six different times before the interface vanishes.

Note: If you find it difficult to treat the Neumann boundary condition here, you may replace the Neumann boundary condition by the Dirichlet boundary condition: \( u|_{\partial \Omega} = 1 \).