Quiz 4

Let \( A = \begin{pmatrix} 1 & s & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \).

(1) Find the value of \( s \) such that \( A \) is diagonalizable.

\textit{Solutions:} The characteristic polynomial is

\[ P_A(\lambda) = \begin{vmatrix} \lambda - 1 & -s & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2) \]

Hence the eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = 1 \) and \( \lambda_3 = 2 \). Since the eigenvalue \( \lambda_1 = 1 \) has multiplicity 2, \( A \) is diagonalizable if and only if the dimension of the 1-eigenspace \( E_1 \) is 2. Note that the \( E_1 \) is given by the solutions of \((I_3 - A)X = 0\), namely,

\[
\begin{pmatrix} 0 & -s & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

We easily see that \( z = 0 \). So if \( s \neq 0 \) then \( y = 0 \) and then \( E_1 \) is just spanned by \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). In this case, the dimension of \( E_1 \) is 1, which is less than the multiplicity 2. Hence \( E_1 \) has dimension 2 if and only if \( s = 0 \), in which case, \( E_1 \) has a basis \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \).

(2) For value \( s \) that \( A \) is diagonalizable, diagonalize \( A \). Namely, find an invertible matrix \( P \) and a diagonal matrix \( \Lambda \) such that \( A = P\Lambda P^{-1} \).

\textit{Solutions:} To diagonalize \( A \), we need find eigenvectors which forms a basis. We have found the basis of \( E_1 \) from the above. It suffices to find a basis of \( E_2 \), which is the space of the solution for the following system (note \( s = 0 \) from the above question):
We easily get a basis $\left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right)$. So we obtain $P = \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right)$ and $\Lambda = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right)$. 