EXTRA CREDIT QUESTION 3

Prove the spectral decomposition: Suppose that $A \in \mathbb{R}^{n \times n}$ is a real symmetric matrix. Then there exists an orthogonal matrix $Q$ such that $A = Q^T \Lambda Q$ with $\Lambda$ a diagonal matrix.

Hint: We can proceed with the following steps:

1. Show that there exists an orthogonal matrix $Q_1$ such that

$$Q_1 A Q_1^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix}$$

where $B$ is an $(n-1) \times (n-1)$-matrix. To find $Q_1$, one needs to put the first column of $Q_1$ to be the unit eigenvector $X_1$ of the eigenvalue $\lambda_1$ (note we have prove in class that $\lambda_1$ and $X_1$ must have real entries). Then we get

$$A = Q_1^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix} Q_1.$$

Since $Q_1$ is orthogonal, we can conclude that $\begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix}$ must be also symmetric and then $* = 0$.

2. Show that $B$ is symmetric.

3. By induction, $B = Q_2^T \Lambda' Q_2$ with $Q_2$ an orthogonal matrix and $\Lambda'$ a diagonal matrix.

4. Show that $Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix} Q_1$ is the required matrix and finish the induction.