Math 265, Practice Midterm 1

Sept 25, 2010

Name: 

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

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1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) If $A^2$ makes sense then $A$ is a square matrix.
(b) A system of linear equations can not have exactly 2 solutions.
(c) If $AB = AC$ and $A \neq 0$ then $B = C$.
(d) Let $W$ be a subspace of a vector space $V$. If $v \in W$ then $-v \in W$.
(e) $\det(2A) = 2 \det(A)$.

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2. Quick Questions, $A, B, C, X, b$ are always matrices here:

(a) Suppose that $\det(A) = \det(A^{-1})$. $\det(A) =$?

Solution: $\det(A) = \det(A^{-1}) = (\det(A))^{-1}$. So $\det(A)^2 = 1$ and then $\det(A) = \pm 1$.

(b) Suppose $AX = 2X$ and $A^3X = aX$. Then $a =$?

Solution:

$A^3X = AAX = AA(2X) = 2AAX = 2A(2X) = 2^2AX = 2^3X$.

So $a = 8$.

(c) The set $\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | x, y \text{ are integers} \}$ is not a subspace of $\mathbb{R}^2$, why?

Solution: The set $S$ defined above is not closed under scalar multiplication: Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ nd $c = 0.5$. We see that $cv = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ is not in $S$. 
3. Let

\[ A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \end{bmatrix}. \]

(a) Compute \(2I_2 - AB^T\).

Solution: \(2I_2 - AB^T = 2I_2 - \begin{bmatrix} -1 & 3 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -7 & 8 \end{bmatrix}\)

(b) Is \(2I_2 - AB^T\) invertible? If it is then find the inverse.

Solution: \(\det(2I_2 - AB^T) = 24 - 21 = 3 \neq 0\). So the matrix is invertible, and the inverse is

\[ \frac{1}{3} \begin{bmatrix} 8 & 3 \\ 7 & 3 \end{bmatrix}. \]
4. Consider the following linear system

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
1 & 1 & a^2 - 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
7 \\
a
\end{pmatrix}
\]

(a) Determine all values of \(a\) such that the system has no solution.

Solution: We can get the echelon form of the augmented matrix as following

\[
\begin{pmatrix}
1 & 1 & 0 & | & 2 \\
0 & 1 & 1 & | & 5 \\
0 & 0 & a^2 - 4 & | & a - 2
\end{pmatrix}
\]

So if the system has no solution if and only if \(a^2 - 4 = 0\) but \(a - 2 \neq 0\).

So \(a = -2\).

(b) Determine all values of \(a\) such that the system has infinitely many solutions.

Solution: In this case, \(a^2 - 4 = a - 2 = 0\). So \(a = 2\).

(c) Determine all values of \(a\) such that the system has a unique solution.

Solution: In this case, \(a^2 - 4 \neq 0\). So \(a \neq \pm 2\).
5. Let 

\[ A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \]

(a) Find \( A^{-1} \).

Solution:

\[ A^{-1} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \]

(b) Compute \( \text{adj}(A) \).

Solution: Since \( \det(A) = 1 \), we have \( \text{adj}(A) = A^{-1} \).
6. Solve the following linear system using Cramer’s rule.

\[-2x_1 + 3x_2 - x_3 = 1\]
\[x_1 + 2x_2 - x_3 = 4\]
\[-2x_1 - x_2 + x_3 = -3\]

Solution: We have the following matrices:

\[A = \begin{pmatrix}
-2 & 3 & -1 \\
1 & 2 & -1 \\
-2 & -1 & 1
\end{pmatrix}\]

\[A_1 = \begin{pmatrix}
1 & 3 & -1 \\
4 & 2 & -1 \\
-3 & -1 & 1
\end{pmatrix}\]

\[A_2 = \begin{pmatrix}
-2 & 1 & -1 \\
1 & 4 & -1 \\
-2 & -3 & 1
\end{pmatrix}\]

\[A_3 = \begin{pmatrix}
-2 & 3 & 1 \\
1 & 2 & 4 \\
-2 & -1 & -3
\end{pmatrix}\]

By Cramer’s rule, we have

\[x_1 = \frac{|A_1|}{|A|} = 2, \quad x_2 = \frac{|A_2|}{|A|} = 3, \quad x_3 = \frac{|A_3|}{|A|} = 4.\]
7. Let \( V \) be the set of all \( 2 \times 2 \) matrices \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that \( abcd = 0 \). Let operation \( \oplus \) be the standard addition of matrices and the operation \( \odot \) be the standard scalar multiplication of matrices.

(a) Is \( V \) closed under addition?
Solution: No, \( V \) is not closed under addition: For an example: Let \( x = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in V \) and \( y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in V \). Then we have \( x + y = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \) is not in \( V \).

(b) Is \( V \) closed under scalar multiplication?
Solution: Yes, \( V \) is closed under scalar multiplication: Suppose \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V \). Note that \( xA = \begin{pmatrix} xa & xb \\ xc & xd \end{pmatrix} \). If \( abcd = 0 \) then we have
\[
(xa)(xb)(xc)(xd) = x^4abcd = 0.
\]
So \( xA \) is in \( V \).

(c) What is zero vector in the set \( V \)?
Solution: The zero vector is \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

(d) Is \( V \) a vector space? Explain.
Solution: No, \( V \) is not a vector space because it is not closed under addition.