Math 265, Practice Midterm 2

Name: ________________________________

Section: __________

This exam consists of 8 pages including this front page.

**Ground Rules**

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

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1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) It is not possible to find 4 linearly independent vectors $v_1, v_2, v_3, v_4$ in $\mathbb{R}^3$.

(b) Let $A$ be an $m \times n$ matrix. If the dimension of the column space of $A$ is 5, then $n \geq 5$.

(c) Consider a system of linear equations $AX = b$ where $A$ is a square matrix. If $\det(A) = 0$ then the system is inconsistent.

(d) Let $V$ be an inner product space and $u, v, w \in V$. If $w$ is orthogonal to $u$ and $v$, then $w$ is orthogonal to any linear combination of $u$ and $v$.

(e) Let $A$ be an $n \times n$-matrix. If $\text{rank}(A) < n$ then $\det(A) = 0$.

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2. Quick Questions, $A$, $B$, $C$, $X$, $b$ are always matrices here:

(a) Suppose $u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find the length of $u$ and $v$.

*Solutions:* $\| u \| = \sqrt{2}$ and $\| v \| = \sqrt{14}$.

(b) Let $u$ and $v$ be as the above question and $\theta$ is the angle between $u$ and $v$. Find $\cos(\theta)$.

*Solutions:*

$$\cos(\theta) = \frac{\langle u, v \rangle}{\| u \| \| v \|} = -\frac{2}{\sqrt{2} \cdot 14} = -\frac{1}{\sqrt{7}}.$$

(c) Find a basis for the space of $3 \times 3$ real symmetric matrices.

*Solutions:*

$$\begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{cases}$$
3. (a) For what values of $c$ are the vectors \[
\begin{pmatrix}
-1 \\
0 \\
-1
\end{pmatrix}
, \quad
\begin{pmatrix}
2 \\
1 \\
2
\end{pmatrix}
\quad\text{and}\quad
\begin{pmatrix}
1 \\
1 \\
c
\end{pmatrix}
\] in $\mathbb{R}^3$ linearly independent?

\textit{Solution:} The three vectors are linearly independent if and only if
\[
\begin{vmatrix}
-1 & 2 & 1 \\
0 & 1 & 1 \\
-1 & 2 & c
\end{vmatrix}
\neq -(c-1) \neq 0.
\]

Hence $c \neq 1$.

(b) If possible, find $a, b, c$ so that \[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\] is orthogonal to \[
\begin{pmatrix}
1 \\
2 \\
1
\end{pmatrix}
\] and \[
\begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}
\].

\textit{Solution:} $u \perp v$ and $u \perp w$ implies that $a + 2b + c = 0$ and $a - b + c = 0$. So $b = 0$ and $a = -c$. Hence $u = c \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}$ for any constant $c$. 

Let \( A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ 2 & -1 & -1 & 0 \end{pmatrix} \).

1. Find a basis for row space of \( A \).
   \( \text{Solutions:} \) We compute the the reduced echelon form of \( A \) is
   \[
   \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
   \]
   Hence the first two rows \((1 \ 0 \ -1 \ 1), (0 \ 1 \ -1 \ 2)\) forms a basis of the row space of \( A \).

2. Find a basis for the column space of \( A \).
   \( \text{Solutions:} \) The first two columns of the reduced echelon form have the leading ones. So the first two columns \( \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \) of \( A \) forms a basis of the column space of \( A \).

3. Find a basis for the null space of \( A \).
   \( \text{Solutions:} \) From the reduced echelon form, we get equations \( x_1 - x_3 + x_4 = 0 \) and \( x_2 - x_3 + 2x_4 = 0 \). Therefore, we have \( x_1 = x_3 - x_4 \) and \( x_2 = x_3 - 2x_4 \). Hence, any vector in \( N(A) \) can be written as
   \[
   \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 - x_4 \\ x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix}
   \]
   with \( x_3, x_4 \) free parameters. So \( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \) forms a basis of the null space.

4. Verify the equality \( \text{rank}(A) + \text{Nullity}(A) = n \).
   \( \text{Solutions:} \) We find that \( \text{rank}(A) = 2 \) from (1) and \( \text{Nullity}(A) = 2 \) from (3). So
   \[
   \text{rank}(A) + \text{Nullity}(A) = 2 + 2 = 4 = n.
   \]
5. Let $V \subset \mathbb{R}^4$ be a subspace spanned by \[
\begin{bmatrix}
1 \\
1 \\
0 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}.
\]

1. Find a orthonormal basis of $V$.

**Solutions:** Let $v_1, v_2$ and $v_3$ denote the above vectors. By Gram-Schmidt process, we compute $w_1 = v_1$ and

\[
\begin{aligned}
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix}
\end{aligned}
\]

For future computation, we replace $w_2$ by $3w_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$. Now

\[
\begin{aligned}
w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
&= v_3 - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
\]

because $\langle v_3, w_1 \rangle = 0$ and $\langle v_3, w_2 \rangle = 0$.

Now we get an orthonormal basis $u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$, and $u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

2. Let $u = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Find $\text{Proj}_V u$.

**Solutions:** By the formula of projection, we have

\[
\begin{aligned}
\text{Proj}_V u &= \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle u, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3 \\
&= \frac{1}{3} w_1 + \frac{1}{15} w_2 + 0 w_3 \\
&= \frac{1}{5} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.
\end{aligned}
\]
6. Consider the following system of linear equation

\[
\begin{align*}
    x_1 + x_2 &= 1 \\
    x_1 + 2x_2 &= 2 \\
    x_1 - x_2 &= 0
\end{align*}
\]

1. Find the rank of the coefficient matrix and the rank of the augmented matrix

*Solutions:* The coefficient matrix is

\[
\begin{pmatrix}
    1 & 1 \\
    1 & 2 \\
    1 & -1
\end{pmatrix}
\]

and the augmented matrix is

\[
\begin{pmatrix}
    1 & 1 & 1 \\
    1 & 2 & 2 \\
    1 & -1 & 0
\end{pmatrix}
\]

The row reduced echelon form of the augmented matrix is

\[
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}
\]

Hence the rank of the coefficient matrix is 2 and the rank of augmented matrix is 3.

2. Is the system consistent? why?

*Solutions:* No, because the rank of coefficient matrix and the rank of augmented matrix are different.

3. If the system is inconsistent. Compute the least squares solution.

*Solutions:* The least square solution satisfies that \( A^TAX = A^Tb \). Hence we get the system of equations

\[
\begin{pmatrix}
    3 & 2 \\
    2 & 6
\end{pmatrix}
\begin{pmatrix}
    \hat{x}_1 \\
    \hat{x}_2
\end{pmatrix} = \begin{pmatrix}
    3 \\
    5
\end{pmatrix}
\]

So \( \hat{x}_1 = \frac{9}{14} \) and \( \hat{x}_2 = \frac{4}{7} \).
7. Let $A$ be a $3 \times 5$-matrix.

1. What will be the maximal possible rank of $A$.
   
   Solutions: The maximal possible rank is 3.

2. Show that columns of $A$ are linearly dependent.
   
   proof: Let $C(A)$ denote the column space of $A$, that is, the space spanned by column vectors of $A$. Since the dimension of $C(A)$ is the rank of $A$, the dimension of $C(A)$ is at most 3. But $A$ has 5 columns vectors and 5 vectors in a 3-dimensional space must be linearly dependent.

3. what will be the maximal possible dimension of $N(A)$.
   
   Solutions: Since $\text{nullity}(A)+\text{rank}(A) = n = 5$, and the minimal possible rank is 0 (in this case, $A$ is necessary a zero matrix). Then the maximal possible dimension of $N(A)$ is 5.