Math 265, Practice Midterm 1
Sept 19, 2012

Name: ________________________________________________

This exam consists of 8 pages including this front page.

**Ground Rules**

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

<table>
<thead>
<tr>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 15</td>
</tr>
<tr>
<td>2 15</td>
</tr>
<tr>
<td>3 15</td>
</tr>
<tr>
<td>4 15</td>
</tr>
<tr>
<td>5 20</td>
</tr>
<tr>
<td>6 10</td>
</tr>
<tr>
<td>7 10</td>
</tr>
</tbody>
</table>

*Total 100*
1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. $A$, $B$, $C$, $X$, $b$ are always matrices here.

(a) If $A^2$ makes sense then $A$ is a square matrix.
(b) A system of linear equations can not have exactly 2 solutions.
(c) If $AB = AC$ and $A \neq 0$ then $B = C$.
(d) Let $W$ be a subspace of a vector space $V$. If $v \in W$ then $-v \in W$.
(e) $\det(2A) = 2 \det(A)$.

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answer</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>
2. Quick Questions, $A$, $B$, $C$, $X$, $b$ are always matrices here:

(a) Suppose that $\det(A) = \det(A^{-1})$. Find $\det(A)$.

\textit{Solutions:} Since $\det(A) = \det(A)^{-1}$, $\det(A)^2 = 1$. So $\det(A) = \pm 1$.

(b) Suppose $AX = 2X$ and $A^3X = aX$. Then $a =$?

\textit{Solutions:} $a = 8$

(c) Is the set $\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | x, y \text{ are integers} \}$ a subspace of $\mathbb{R}^2$, why?

\textit{Solutions:} Let $c = 0.5$ and $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $cu = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ is not in the set. Hence the set is not closed under scalar multiplication and it is \textit{not} a subspace of $\mathbb{R}^2$. 

3
3. Let 
\[ A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \end{bmatrix}. \]

(a) Compute \(2I_2 - AB^T\).

\[ 2I_2 - AB^T = \begin{bmatrix} 3 & -3 \\ -7 & 8 \end{bmatrix} \]

(b) Is \(2I_2 - AB^T\) invertible? If it is then find the inverse.

\[ 2I_2 - AB^T = \frac{1}{3} \begin{bmatrix} 8 & 3 \\ 7 & 3 \end{bmatrix} \]
4. Consider the following linear system

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
1 & 1 & a^2 - 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
2 \\
7 \\
a
\end{pmatrix}
\]

(a) Determine all values of \(a\) such that the system has no solution.

\textit{Solutions:} The augmented matrix of the above system is

\[
\begin{pmatrix}
1 & 1 & 0 & 2 \\
1 & 2 & 1 & 7 \\
1 & 1 & a^2 - 4 & a
\end{pmatrix}
\]

Then we get the echelon form:

\[
\begin{pmatrix}
1 & 1 & 0 & 2 \\
0 & 1 & 0 & 5 \\
0 & 0 & a^2 - 4 & a - 2
\end{pmatrix}
\]

Then the system has no solution if and only if \(a^2 - 4 = 0\) and \(a - 2 \neq 0\).

So \(a = -2\)

(b) Determine all values of \(a\) such that the system has infinitely many solutions.

\textit{Solutions:} The system has infinitely many solutions if and only if \(a^2 - 4 = a - 2 = 0\).

So \(a = 2\).

(c) Determine all values of \(a\) such that the system has a unique solution.

\textit{Solutions:} The system has unique solution if and only if \(a^2 - 4 \neq 0\).

That is \(a \neq \pm 2\).
5. (a) Compute
\[
\begin{bmatrix}
-2 & 0 & 0 & 0 \\
5 & 3 & 5 & 7 \\
3 & 0 & 2 & 1 \\
8 & 0 & 2 & 2 \\
\end{bmatrix}
\]

Solution: By cofactor’s formula, we get
\[
\det(A) = -2 \begin{vmatrix} 3 & 5 & 7 \\
0 & 2 & 1 \\
0 & 2 & 2 \\
\end{vmatrix} = (-2)3 \begin{vmatrix} 2 & 1 \\
2 & 2 \\
\end{vmatrix} = -2 \cdot 3 \cdot 2 = -12.
\]

(b) Compute \(A^{-1}\), where

\[
A = \begin{bmatrix} 1 & 0 & 2 \\
0 & 1 & 2 \\
2 & 1 & 5 \\
\end{bmatrix}
\]

Solution:
\[
A^{-1} = \begin{bmatrix} -3 & -2 & 2 \\
-4 & -1 & 2 \\
2 & 1 & -1 \\
\end{bmatrix}
\]
6. Solve the following linear system using Cramer’s rule.

\[-2x_1 + 3x_2 - x_3 = 1\]
\[x_1 + 2x_2 - x_3 = 4\]
\[-2x_1 - x_2 + x_3 = -3\]

Solutions: By Cramer’s rule.

\[
|A| = \begin{vmatrix}
-2 & 3 & -1 \\
1 & 2 & -1 \\
-2 & -1 & 1
\end{vmatrix} = -2
\]

\[
|A_1| = \begin{vmatrix}
1 & 3 & -1 \\
4 & 2 & -1 \\
-3 & -1 & 1
\end{vmatrix} = -4
\]

\[
|A_2| = \begin{vmatrix}
-2 & 1 & -1 \\
1 & 4 & -1 \\
-2 & -3 & 1
\end{vmatrix} = -6
\]

\[
|A_3| = \begin{vmatrix}
-2 & 3 & 1 \\
1 & 2 & 4 \\
-2 & -1 & -3
\end{vmatrix} = -8
\]

So
\[x_1 = \frac{|A_1|}{|A|} = 2, \quad x_2 = \frac{|A_2|}{|A|} = 3, \quad x_3 = \frac{|A_3|}{|A|} = 4.\]
7. Let $V$ be the set of all $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $abcd = 0$. Let operation $\oplus$ be the standard addition of matrices and the operation $\odot$ be the standard scalar multiplication of matrices.

(a) Is $V$ closed under addition?

Solution: No. Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Both $A$ and $B$ are in $V$. But $A + B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is not in $V$.

(b) Is $V$ closed under scalar multiplication?

Solution: Yes. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $V$. By definition $abcd = 0$. Then for any $x \in \mathbb{R}$, $xA = \begin{pmatrix} xa & ab \\ xc & xd \end{pmatrix}$. We get $xaxbxcxd = x^4abcd = 0$. So $cA$ is in $V$.

(c) What is zero vector in the set $V$?

Solutions: It is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(d) Is $V$ a vector space? Explain.

Solutions: No, it is not a vector space because it is not closed under addition.