Lyubeznik numbers of projective schemes

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• For any Noetherian commutative ring $R$ and an ideal $I$ of $R$, one can define a functor $\Gamma_I$ as

$$\Gamma_I(M) = \{x \in M | I^n x = 0 \text{ for some integer } n\}$$

for any $R$-module $M$.

• $H^i_I(M) = \mathcal{R}^i \Gamma_I(M)$.

• also $H^i_I(M) = \lim_{\longrightarrow} \text{Ext}^i_R(A/I^n, M)$.
• If \((R, m)\) is a regular local ring containing a field, the following properties are known (Huneke-Sharp; Lyubeznik)

1. \(\text{Ass}_{R}(H^{i}_{I}(R))\) is finite for all \(i\);

2. the Bass numbers of \(H^{i}_{I}(R)\) are finite for all \(i\).

3. \(H^{i}_{m}(H^{j}_{I}(R))\) are injective.

Remark: to prove this result in char. 0, one has to use D-module theory!
• If $A$ is a local ring containing a field $k$ and admits a surjection $R \to A$ where $(R, m)$ is a $n$-dim regular local ring containing $k$, then one can define the Lyubeznik numbers

$$\lambda_{i,j}(A) := \dim_k(\text{Ext}_R^i(R/m, H_I^{n-j}(R))).$$

• $\lambda_{i,j}(A)$ do NOT depend on the choice of $R \to A$ (Lyubeznik’93).

• If $A$ is a local ring containing a field $k$, then one can define (due to Lyubeznik’93)

$$\lambda_{i,j}(A) := \lambda_{i,j}(\hat{A}).$$
• $\lambda_{i,j}(A)$ are finite (cf. 2nd slide).

• $H_m^i(H_{I}^{n-j}(R)) \cong E^{\lambda_{i,j}(A)}$ (due to Lyubeznik)

• By the highest Lyubeznik number, we mean $\lambda_{d,d}(A)$, $d = \dim(A)$. 
Let $X$ be a projective scheme over a field $k$ (assume $k = \bar{k}$). Given an embedding $\eta : X \to \mathbb{P}^n_k$, one can write $X = \text{Porj}(k[x_0, \ldots, x_n]/I)$, where $I$ is a homogeneous ideal. Let $A = (k[x_0, \ldots, x_n]/I)_{(x_0, \ldots, x_n)}$. Then one can consider the Lyubeznik numbers of $A$.

In 2007, it is proven (by myself) that the highest Lyubeznik number of $A$ is a numerical invariant of $X$, i.e., it depends only on $X$ itself, but NOT on the embedding, which provides supporting evidence to a positive answer to the following question:

**Question:** With notations as above, is it true that all $\lambda_{i,j}(A)$ depend only on $X$, but not on the embedding?
Why interesting?

Short Answer: connection with topology!

Example (essentially due to Garcia-Lopéz and Sabbah): Let $X$ be a smooth complex projective variety. Then

$$\lambda_{0,j+1}(A) = b_j(X),$$

where $b_j(X)$ is the $j$-th Betti number of $X$, and other $\lambda_{i,j}(A)$ can be determined by $\lambda_{0,j}(A)$s.

Remark. If the variety $X$ in the above example is singular, then we can not say anything about those numbers. However, if $\text{char}(k) = p > 0$, then we have the following
**Main Theorem** When $\text{char}(k) = p > 0$, each $\lambda_{i,j}(A)$ can only achieve finitely many possible values for all choices of embeddings.

The proof of this result is based on (or inspired by) Lyubeznik’s F-module theory.
Before we can outline the proof, let’s introduce some notations.

- $R = k[x_0, \ldots, x_n]$, $I$ is the defining ideal of the projective scheme $X$. Since field extensions do not change Lyubeznik numbers, we assume $k = \overline{k}$. Let

$$\mathcal{M} = \text{Ext}^{n+1-i}_R(\text{Ext}^{n+1-j}_R(R/I, R), R).$$

- Let $\{L_i, \theta_{ij}\}$ be an inverse system of $R$-modules and assume that $L_i$ are graded and all $\theta_{ij}$ are degree-preserving. Then define $\lim_{\leftarrow i} L_i$ as follows

$$\left(\lim_{\leftarrow i} L_i\right)_l = \lim_{\leftarrow i}(L_i)_l$$
$\mathcal{M}$ is a very interesting object.

- $\mathcal{M}$ is naturally graded and its degree-0 piece only depend on $X$ but not on the embedding.

**Reason.** When $i \geq 2$,

\[
\mathcal{M}_0 \cong \text{Hom}_k(H^{i-1}(X, \text{Ext}^{n+1-j}(\mathcal{O}_X, \omega_{\mathbb{P}^n})), k)
\]

where $\text{Ext}^{n+1-j}(\mathcal{O}_X, \omega_{\mathbb{P}^n})$ depends only on $X$ since it is the $(-j)$-th cohomology sheaf of the dualizing complex on $X$. The proof of the case $i \leq 1$ is done by considering some exact sequences, which will be skipped here.
There is a natural action of Frobenius (or a $p$-linear endomorphism) on $\mathcal{M}$.

$$\mathcal{M} \xrightarrow{\alpha} R^{(1)} \otimes_R \mathcal{M}$$

$$\xrightarrow{\beta} \text{Ext}_{R}^{n+1-i}(\text{Ext}_{R}^{n+1-j}(R/I[p], R), R)$$

$$\xrightarrow{\gamma} \mathcal{M}$$

where $\alpha(m) = 1 \otimes m$, $\beta$ is the natural isomorphism, and $\gamma$ is induced by $R/I[p] \to R/I$. Then the action of Frobenius $f : \mathcal{M} \to \mathcal{M}$ is defined to be $\gamma \circ \beta \circ \alpha$, noticing that $\beta$ and $\gamma$ are $R$-linear and $\alpha$ is $p$-linear.

An important feature of $f$: $\text{deg}(f(m)) = p \text{deg}(m)$, for all homogeneous $m \in \mathcal{M}$. 

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Once we have such an action of Frobenius on $\mathcal{M}$, we can consider

$$\mathcal{M}_s := \bigcap_e (f^e(\mathcal{M}))$$

called the stable part of $\mathcal{M}$.

**Theorem**

1. $\mathcal{M}_s \subseteq \mathcal{M}_0$ and is a finite-dimensional $k$-space.

2. $\dim_k(\mathcal{M}_s) = \lambda_{i,j}(A)$
The first part of the above theorem is fairly easy. To prove the second part, let $\mathcal{N}$ be the $R$-submodule of $\mathcal{M}$ generated by $\mathcal{M}_s$, and then prove the following:

1. $\displaystyle \lim_{e} F^e(\mathcal{N}) \cong \lim_{e} F^e(\mathcal{M})$

2. $\displaystyle \lim_{e} F^e(\mathcal{N}) \cong R^{\dim_k(\mathcal{M}_s)}$

3. $\displaystyle \lim_{e} F^e(\mathcal{M}) \cong R^{\lambda_{i,j}}(A)$

where $F$ is the Frobenius on $R$. 

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From what we have seen, one can notice that actually we are very close to a complete solution. Namely, if we can show that this action of Frobenius restricted to $\mathcal{M}_0$ does not depend on the embedding, then it follows that $\lambda_{i,j}(A)$ do not depend on the embedding. We believe this should be the case and we pose it here as a conjecture:

**Conjecture.** With notations as above, the action of Frobenius $f : \mathcal{M} \to \mathcal{M}$ restricted to $\mathcal{M}_0$ does not depend on the embedding.