Stiffness and Mass Matrices

Yingwei Wang

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Outline

Basis functions
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  Formula

Stiffness Matrix
  General case
  Special case

Mass Matrix
  General case
  Special case

Remarks
1.1 Basis functions

- Linear basis functions on $[0, 1]$

Figure 1: Piecewise polynomials with degree one
1.2 Basis functions—Formula

Function
Let $0 = x_0 < x_1 < \cdots < x_n = 1$, $h_i = x_i - x_{i-1}$, then the basis functions $\{\phi_i(x)\}_{i=1}^{n-1}$ are defined as

$$\phi_i(x) = \begin{cases} \frac{1}{h_i} (x - x_{i-1}), & x \in [x_{i-1}, x_i), \\ -\frac{1}{h_{i+1}} (x - x_{i+1}), & x \in [x_i, x_{i+1}). \\ 0, & \text{otherwise} \end{cases}$$

Derivative
It is easy to verify that the first derivative of basis functions are

$$\phi_i'(x) = \begin{cases} \frac{1}{h_i}, & x \in (x_{i-1}, x_i), \\ -\frac{1}{h_{i+1}}, & x \in (x_i, x_{i+1}). \\ 0, & \text{otherwise} \end{cases}$$
2.1 Stiffness Matrix — General case

Definition The stiffness matrix $K = (k_{ij})$ is defined by

$$k_{ij} = a(\phi_i, \phi_j) = \int_0^1 \phi'_i(x)\phi'_j(x) dx$$

Formula The matrix entries $k_{ij}$ can be easily calculated as follows:

$$k_{ii} = \int_{x_{i-1}}^{x_{i+1}} (\phi'_i)^2 dx = \int_{x_{i-1}}^{x_i} \frac{1}{h_i^2} dx + \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}^2} dx$$

$$= h_i^{-1} + h_{i+1}^{-1},$$

$$k_{i,i+1} = k_{i+1,i} = \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}} \left(- \frac{1}{h_{i+1}}\right) dx$$

$$= -h_{i+1}^{-1},$$

$$k_{i,j} = k_{j,i} = 0, \quad j > i + 1.$$
2.2 Stiffness Matrix—Special case

- Equally-spaced points: Let $h_i = h$, $i = 1, \ldots, n-1$, then the stiffness matrix becomes

$$K = \frac{1}{h} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdots & \cdots & \cdots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$
§3.1 Mass Matrix—General case

▶ Definition The Mass matrix $M = (m_{ij})$ is defined by

$$m_{ij} = (\phi_i, \phi_j) = \int_0^1 \phi_i(x)\phi_j(x)dx$$

▶ Formula The matrix entries $m_{ij}$ are

$$m_{ii} = \int_{x_{i-1}}^{x_{i+1}} (\phi_i)^2 dx$$

$$= \frac{1}{h_i^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 dx + \frac{1}{h_{i+1}^2} \int_{x_i}^{x_{i+1}} (x - x_{i+1})^2 dx$$

$$= \frac{1}{3} (h_i + h_{i+1}),$$

$$m_{i,i+1} = m_{i+1,i} = \frac{1}{h_{i+1}^2} \int_{x_i}^{x_{i+1}} (x - x_i)(x_{i+1} - x) dx$$

$$= \frac{1}{6} h_{i+1},$$

$$m_{i,j} = m_{j,i} = 0, \quad j > i + 1.$$
3.2 Mass Matrix—Special case

- **Equally-spaced points** Let \( h_i = h, \ i = 1, \ldots, n - 1 \), then the mass matrix becomes

\[
M = \frac{h}{6} \begin{pmatrix}
4 & 1 \\
1 & 4 & 1 \\
& \cdots & \cdots \\
& 1 & 4 & 1 \\
& 1 & 4
\end{pmatrix}
\]
In Exercises 0.x.1, we are told that if \( f(x) = \sum_{i=1}^{n} f_i \phi_i(x) \), then the inner product \( (f, \phi_i) \) can be computed by mass matrix. However, mass matrix is usually employed to solve the problems like

\[-\Delta u + \alpha u = f.\]
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Stiffness matrix in 1-dim with equally-spaced points is similar as the differential matrix in finite differences
\[ u_j'' \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}. \]

What if in the 2-dim or 3-dim cases?
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What if in the 2-dim or 3-dim cases?

How about the piecewise quadratic basis functions? My friends will show that. (0.x.4)
Thanks!