Spaces of Null Homotopic Maps

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Abstract. We study the null component of the space of pointed maps from $B\pi$ to $X$ when $\pi$ is a locally finite group, and other components of the mapping space when $\pi$ is elementary abelian. Results about the null component are used to give a general criterion for the existence of torsion in arbitrarily high dimensions in the homotopy of $X$.

§1. Introduction

In 1983 Haynes Miller [M] proved a conjecture of Sullivan and used it to show that if $\pi$ is a locally finite group and $X$ is a simply connected finite dimensional CW-complex then the space of pointed maps from the classifying space $B\pi$ to $X$ is weakly contractible, i.e. $\text{Map}_*(B\pi, X) \simeq *$. This result had immediate applications. Alex Zabrodsky [Z] used it to study maps between classifying spaces of compact Lie groups. McGibbon and Neisendorfer [MN] applied Miller’s theorem to answer a question of Serre; they proved that if $X$ is a simply connected finite dimensional CW-complex with $H^*(X, F_p) \neq 0$ then there are infinitely many dimensions in which $\pi_*(X)$ has $p$-torsion.

The goal of this note is to use the functor $T^V$ of [L] to generalize Miller’s theorem and some of its corollaries to a large class of infinite dimensional spaces (see [LS2] for closely related earlier work in this direction). This generalization comes at the expense of working with one component of the function complex $\text{Map}_*(B\pi, X)$ at a time.

Fix a prime number $p$.

Theorem 1.1. Let $\pi$ be a locally finite group and $X$ a simply connected $p$-complete space. Assume that $H^*(X, F_p)$ is finitely generated as an algebra. Then the component of $\text{Map}_*(B\pi, X)$ which contains the constant map is weakly contractible.

Remark: There is a standard way [M, 1.5] to relax the assumption in 1.1 that $X$ is $p$-complete.

Theorem 1.1 is actually a special case of a more general assertion. Recall that an unstable module $M$ over the mod $p$ Steenrod Algebra $A_p$ is said to be locally finite [LS] if each element $x \in M$ is contained in a finite $A_p$ submodule. If $R$ is a connected unstable algebra over $A_p$ then the augmentation ideal $I(R)$ is by definition the ideal of positive-dimensional elements and the module of indecomposables $Q(R)$ is the unstable $A_p$ module $I(R)/I(R)^2$. An unstable algebra $R$ over $A_p$ is of finite type if each $R^k$ is finite-dimensional as an $F_p$ vector space.

Theorem 1.2. Let $\pi$ be a locally finite group and $X$ a simply connected $p$-complete space such that $H^*(X, F_p)$ is of finite type. Assume that the module of indecomposables $Q(H^*(X, F_p))$ is locally finite as a module over $A_p$. Then the component of $\text{Map}_*(B\pi, X)$ which contains the constant map is weakly contractible.

Remark: Theorem 1.1 does in fact follow from Theorem 1.2, since if $H^*(X, F_p)$ is finitely generated as an algebra then $Q(H^*(X, F_p))$ is a finite $A_p$ module.

Remark: Theorem 1.2 has a converse, at least if $p = 2$ (see Theorem 3.2). There is also a generalization of 1.2 that deals with other components of the mapping space $\text{Map}_*(B\pi, X)$ (see Theorem 4.1) but for this generalization it is necessary to assume that $\pi$ is an elementary abelian $p$-group.

Given 1.2, the arguments of [MN] go over more or less directly and lead to the following result. A CW-complex is of finite type if it has a finite number of cells in each dimension.
THEOREM 1.3. Suppose that $X$ is a two-connected CW-complex of finite type. Assume that $H^*(X, F_p) \neq 0$ and that $Q(H^*(X, F_p))$ is locally finite as a module over $A_p$. Then there exist infinitely many $k$ such that $\pi_k(X)$ has $p$-torsion.

REMARK: The example of $CP^\infty$ shows that it would not be enough in Theorem 1.3 to assume that $X$ is 1-connected.

Some instances of 1.3 were previously known; for instance, if $X = BG$ for $G$ a suitable compact Lie group then the conclusion of 1.3 can be obtained by applying [MN] to the loop space on $X$. However, Theorem 1.3 applies in many previously inaccessible cases; for example, it applies if $X$ is the Borel construction $EG \times_G Y$ of the action of a compact Lie group $G$ on a finite complex $Y$ or if $X$ is a quotient space obtained from such a Borel construction by collapsing out a skeleton.

We first noticed Theorem 1.1 as part of our work [DW] on calculating fragments of $T^V$ with Smith theory techniques. The proof of 1.1 given here does not use the localization approach of [DW]; it is partly for this reason that the proof generalizes to give 1.2.

Organization of the paper. Section 2 recalls some properties of the functor $T^V$. In §3 there is a proof of 1.2 and in §4 a generalization of 1.2 to other components of the mapping space. Section 5 uses the ideas of [MN] to deduce 1.3 from 1.2.

Notation and terminology. The prime $p$ is fixed for the rest of the paper; all unspecified cohomology is taken with $F_p$ coefficients. The symbol $\mathcal{U}$ (resp. $\mathcal{K}$) will denote the category of unstable modules (resp. algebras) $[L]$ over $A_p$. If $R \in \mathcal{K}$ then $U(R)$ (resp. $K(R)$) will stand for the category of objects of $\mathcal{U}$ (resp. $\mathcal{K}$) which are also $R$-modules (resp. $R$-algebras) in a compatible way [DW].

For a pointed map $f : K \to X$ of spaces we will let $\text{Map}_*(K, X)_f$ denote the component of the pointed mapping space $\text{Map}_*(K, X)$ containing $f$. The component of the unpointed mapping space containing $f$ is $\text{Map}(K, X)_f$.

§2 The functor $T^V$

Let $V$ be an elementary abelian $p$-group, i.e., a finite-dimensional vector space over $F_p$, and $H^V$ the classifying space cohomology $H^*BV$. Lannes $[L]$ has constructed a functor $T^V : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor given by tensor product (over $F_p$) with $H^V$ and has shown that $T^V$ lifts to a functor $\mathcal{K} \to \mathcal{K}$ which is similarly left adjoint to tensoring with $H^V$.

**Proposition 2.1 [L].** For any object $R$ of $\mathcal{K}$ the functor $T^V$ induces functors $U(R) \to U(T^V(R))$ and $K(R) \to K(T^V(R))$. The functor $T^V$ is exact, and preserves tensor products in the sense that if $M$ and $N$ are objects of $U(R)$ there is a natural isomorphism

$$T^V(M \otimes_R N) \cong T^V(M) \otimes_{T^V(R)} T^V(N)$$

Now suppose that $\gamma : R \to H^V$ is a $\mathcal{K}$-map. The adjoint of $\gamma$ is a map $T^V(R) \to F_p$ or in other words a ring homomorphism $\hat{\gamma} : T^V(R)^0 \to F_p$. For $M \in U(R)$, let $T^V_\gamma(M)$ be the tensor product $T^V(M) \otimes_{T^V(R)^0} F_p$, where the action of $T^V(R)^0$ on $F_p$ is given by $\hat{\gamma}$. Note that $T^V_\gamma(R) \in \mathcal{K}$.

**Proposition 2.2 [DW, 2.1].** For any $\mathcal{K}$-map $\gamma : R \to H^V$ the functor $T^V_\gamma(-)$ induces functors $U(R) \to U(T^V_\gamma(R))$ and $K(R) \to K(T^V_\gamma(R))$. The functor $T^V_\gamma$ is exact, and preserves tensor products in the sense that if $M$ and $N$ are objects of $U(R)$ there is a natural isomorphism

$$T^V_\gamma(M \otimes_R N) \cong T^V_\gamma(M) \otimes_{T^V_\gamma(R)} T^V_\gamma(N)$$

The following proposition is a straightforward consequence of the above two.
LEMMA 2.3. Suppose that $\alpha : R_1 \to R_2$ and $\beta : R_2 \to H^V$ are morphisms of $K$, and let $\gamma : R_1 \to H^V$ denote the composite $\beta \cdot \alpha$.

1. If $\alpha$ is a surjection and $M \in U(R_2)$ is treated via $\alpha$ as an object of $U(R_1)$, then the natural map $T^V(M) \to T^V_\beta(M)$ is an isomorphism.
2. If $M \in U(R_1)$ then the natural map $T^V_\beta(R_2) \otimes T^V_\gamma(M) \to T^V_\beta(R_2 \otimes R_1, M)$ is an isomorphism.

There is a natural map $\lambda_X : T^V(H^*X) \to H^*\text{Map}(BV, X)$ for any space $X$. If $g : BV \to X$ is a map which induces the cohomology homomorphism $\gamma : H^*X \to H^V$ then $\lambda_X$ passes to a quotient map

$$\lambda_{X,g} : T^V_\gamma(H^*X) \to H^*\text{Map}(BV, X)_g.$$ A lot of the geometric usefulness of $T^V$ is explained by the following theorem.

THEOREM 2.4 [L2]. Let $X$ be a 1-connected space, $g : BV \to X$ a map, and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Assume that $H^*X$ is of finite type, that $T^V_\gamma H^*X$ is of finite type, and that $T^V_\gamma H^*X$ vanishes in dimension 1. Then $\lambda_{X,g}$ is an isomorphism.

For any object $M$ of $U$ the adjunction map $M \to H^V \otimes_{F_p} T^V(M)$ can be combined with the unique algebra map $H^V \to F_p$ to give a map $M \to T^V(M)$; call this map $\epsilon$. (If $M = H^*X$ for some space $X$, then $\epsilon$ fits into a commutative diagram involving $\lambda_X$ and the cohomology homomorphism induced by the basepoint evaluation map $\text{Map}(BV, X) \to X$.)

THEOREM 2.5 [LS, 6.3.2]. The map $\epsilon : M \to T^V(M)$ is an isomorphism iff $M$ is locally finite as a module over $A_p$.

If $R \in K$, $M \in U(R)$ and $\gamma : R \to H^V$ is a $K$-map, we will denote the composite $M \to T^V(M) \to T^V_\gamma(M)$ by $\epsilon_\gamma$. Theorem 2.5 leads to the following result, which we will need in §4.

PROPOSITION 2.6. Let $M$ be an object of $U(H^V)$ and $\iota : H^V \to H^V$ the identity map. Then $\epsilon_\iota : M \to T^V_\iota(M)$ is an isomorphism iff $M$ splits as a tensor product $H^V \otimes_{F_p} N$ for some $N \in U$ which is locally finite as a module over $A_p$.

PROOF: The fact that $\epsilon_\iota$ is an isomorphism if $M$ has the stated tensor product decompositon follows directly from 2.3(2), 2.5 and [L, 4.2]. Conversely, under the assumption that $\epsilon_\iota$ is an isomorphism Proposition 2.4 of [DW] guarantees that $M$ splits as a tensor product $H^V \otimes_{F_p} N$ for some $N \in U$; the fact that $N$ is locally finite is again a consequence of 2.3(2) and 2.5.

§3 The null component

In this section we will prove Theorem 1.2. Before doing this we will recast the conclusion of the theorem in a slightly different form.

LEMMA 3.1. Let $K$ be a finite pointed CW-complex, $X$ a 1-connected space, and $f : K \to X$ a pointed map. Then $\text{Map}_*(K, X)_f$ is weakly contractible if and only if the inclusion of the basepoint in $K$ induces a weak equivalence $\text{Map}(K, X)_f \to X$.

PROOF: As in [M, 9.1] the inclusion $\ast \to K$ gives rise to a fibration sequence $\text{Map}_*(K, X)_f \to \text{Map}(K, X)_f \to X$.

The arguments of [M, §9] now show that Theorem 1.2 follows directly from the following result.

THEOREM 3.2. Let $V$ be an elementary abelian $p$-group and $X$ a 1-connected $p$-complete space such that $H^*X$ is of finite type. Let $f : BV \to X$ be a constant map and $\phi : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:

1. $QH^*X$ is locally finite as an $A_p$ module
2. the map $\epsilon_\phi : H^*X \to T^V_\phi H^*X$ is an isomorphism
3. the inclusion of the basepoint $\ast \to BV$ induces a weak equivalence $\text{Map}(BV, X)_f \to X$. 

Let \( \mathrm{Lemma} \ 4.3 \).

As in the case of \( \mathrm{Theorem} \ 3.2 \), it is likely that the three conditions of \( \mathrm{Theorem} \ 4.1 \) are equivalent for any prime \( p \); the proof would depend on the odd primary version of the results in [S].

**Remark 3.3:** It is likely that the three conditions of \( \mathrm{Theorem} \ 1.2 \) are equivalent for any prime \( p \); the proof would depend on the odd primary version of the results in [S].

**Proof of 3.2:** First consider the implication \( (1) \implies (2) \). Let \( R = H^*X \) and let \( I \subseteq R \) be the augmentation ideal. Pick \( s \geq 0 \). The fact that the action of \( R \) on \( I^s/I^{s+1} \) factors through the augmentation \( R \to \mathbb{F}_p \) implies that the action of \( T^V(R) \) on \( T^V(I^s/I^{s+1}) \) factors through the map \( T^V(R) \to T^V(\mathbb{F}_p) \cong \mathbb{F}_p \) induced by augmentation; since this last map is adjoint to \( \phi : R \to H^*(BV) \) it follows from 2.3(1) that the quotient map \( T^V(I^s/I^{s+1}) \to T^V(I^s/I^{s+1}) \) is an isomorphism. Moreover, \( I^s/I^{s+1} \), as a quotient of \((I/I^2)^{\otimes s}\), is the union of its finite \( A_p \) submodules so by 2.5 the map \( \epsilon : I^s/I^{s+1} \to T^V(I^s/I^{s+1}) \) is an isomorphism. Putting these two facts together shows that \( \epsilon_{\phi} : I^s/I^{s+1} \to T^V_{\phi}(I^s/I^{s+1}) \) is an isomorphism. By induction and exactness, then, the map \( \epsilon_{\phi} : R/I^{s+1} \to T^V_{\phi}(R/I^{s+1}) \) is an isomorphism. The map \( T^V(\mathbb{F}_p) \to T^V_{\phi}(\mathbb{F}_p) \cong \mathbb{F}_p \) induced by augmentation is an epimorphism, so by exactness \( T^V_{\phi}(I) \) vanishes in dimension 0. By Lemma 2.2 and exactness, \( T^V_{\phi}(I^{s+1}) \) vanishes up to and including dimension \( s \), and hence again by exactness the map \( T^V(R) \to T^V_{\phi}(R/I^{s+1}) \) induced by the quotient projection \( R \to R/I^{s+1} \) is an isomorphism up through dimension \( s \). It follows immediately that \( \epsilon_{\phi} : R \to T^V(\mathbb{F}_p) \) is an isomorphism.

The implication \( (2) \implies (3) \) is an easy consequence of Theorem 2.4.

For \( (3) \implies (1) \), assume \( p = 2 \). According to [S, proof of 3.1] condition (3) implies that the loop space cohomology \( H^*(\Omega X) \) is locally finite as an \( A_p \) module, i.e., in the terminology of [S], that \( H^*(\Omega X) \in \mathcal{N}l_k \) for all \( k \). According to [S, 2.1(ii)], this implies that \( \Sigma^{-1}QH^*X \in \mathcal{N}l_k \) for all \( k \). This amounts to the assertion that \( \Sigma^{-1}QH^*X \) (or equivalently \( QH^*X \)) is locally finite [S, proof of 3.1].

### §4 Other mapping space components

In this section we will give a generalization of \( \mathrm{Theorem} \ 1.2 \) to mapping space components other than the component containing the constant map; this generalization is limited, however, in that it deals with elementary abelian \( p \)-groups rather than with arbitrary locally finite groups.

Given an elementary abelian \( p \)-group \( V \), call an object \( M \) of \( \mathcal{U}(H^V) \) \( f \)-split if \( M \) is isomorphic to \( H^V \otimes_{\mathbb{F}_p} N \) for some \( N \in \mathcal{U} \) which is locally finite as a module over \( A_p \). Suppose that \( \gamma : R \to H^V \) is a map in \( \mathcal{K} \) with image \( S \subset H^V \) and kernel \( I \subseteq R \). Say that \( \gamma \) is almost \( f \)-split if

(i) \( S \) is a Hopf subalgebra of \( H^V \), and

(ii) for each \( s \geq 0 \) the tensor product \( H^V \otimes_S (I^s/I^{s+1}) \) is \( f \)-split as an object of \( \mathcal{U}(H^V) \).

Recall from 3.1 that \( \mathrm{Map}_p(K,X)_f \) is weakly contractible iff evaluation at the basepoint gives an equivalence \( \mathrm{Map}(K,X)_f \cong X \).

**Theorem 4.1.** Let \( V \) be an elementary abelian \( p \)-group and \( X \) a 1-connected \( p \)-complete space such that \( H^*X \) is of finite type. Let \( g : BV \to X \) be a map and \( \gamma : H^*X \to H^V \) the induced cohomology homomorphism. Consider the following three conditions:

1. \( \gamma \) is almost \( f \)-split
2. the map \( \epsilon : H^*X \to T^V_{\gamma} H^*X \) is an isomorphism
3. the inclusion of the basepoint \( * \to BV \) induces a weak equivalence \( \mathrm{Map}(BV,X)_g \to X \).

Then \( (1) \implies (2) \implies (3) \). Moreover, if \( p = 2 \) then \( (3) \implies (2) \implies (1) \).

**Remark 4.2.** As in the case of \( \mathrm{Theorem} \ 3.2 \), it is likely that the three conditions of \( \mathrm{Theorem} \ 4.1 \) are equivalent for any prime \( p \).

**Lemma 4.3.** Let \( K \) be a pointed \( CW \)-complex, \( X \) a pointed 0-connected space, \( g : K \to X \) a map, and \( f : K \times X \to X \) a constant map. Assume that there exists a map \( m : K \times X \to X \) which is \( 1_X \) on the axis \(* \times X \) and \( g : K \to X \) on the axis \( K \times * \). Then the basepoint evaluation
map $e_f : \text{Map}(K, X)_f \to X$ is a weak equivalence if and only if the corresponding map $e_g : \text{Map}(K, X)_g \to X$ is a weak equivalence.

**Proof:** Construct a commutative diagram

$$
\begin{array}{ccc}
K & \xrightarrow{=} & K \\
\downarrow{a} & & \downarrow{b} \\
K \times X & \xrightarrow{(pr_1, m)} & K \times X
\end{array}
$$

in which $a(k) = (k, *)$, $b(k) = (k, g(k))$ and $pr_1$ is projection on the first factor. Since the lower horizontal map is a weak equivalence, it follows that the induced map $e : \text{Map}(K, K \times X)_a \to \text{Map}(K, K \times X)_b$ is a weak equivalence. It is clear that $e$ commutes with the natural projections from its domain and range to $\text{Map}(K, K)_i$, where $i$ is the identity map of $K$. The lemma follows from the fact that the domain of $e$ is $\text{Map}(K, K)_i \times \text{Map}(K, X)_f$ while the range is $\text{Map}(K, K)_i \times \text{Map}(K, X)_g$.

**Lemma 4.4.** Let $K$ be a pointed CW-complex, $X$ a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that the basepoint evaluation map $e_g : \text{Map}(K, X)_g \to X$ is a weak equivalence. Then the basepoint evaluation map $e_f : \text{Map}(K, X)_f \to X$ is also a weak equivalence.

**Proof:** The map $m$ required in 4.3 is provided up to weak equivalence by the evaluation map $K \times \text{Map}(K, X)_g \to X$.

**Lemma 4.5.** Let $V$ be an elementary abelian $p$-group, $R$ a connected object of $K$, $\gamma : R \to H^V$ a map, and $\phi : R \to H^V$ the trivial map (ie. the map which factors through the augmentation $R \to \mathbb{F}_p$). Assume there exists a map $\mu : R \to H^V \otimes \mathbb{F}_p R$ which gives $1_R$ when combined with the augmentation map of $H^V$ and $\gamma : R \to H^V$ when combined with the augmentation map of $R$. Then $\epsilon : R \to T^V_\gamma (R)$ is an isomorphism if and only if $\epsilon : R \to T^V_\gamma (R)$ is an isomorphism.

**Proof:** This is essentially the proof of 4.3 with the arrows reversed. Construct a commutative diagram

$$
\begin{array}{ccc}
H^V & \xrightarrow{=} & H^V \\
\downarrow{\alpha} & & \downarrow{\beta} \\
H^V \otimes \mathbb{F}_p R & \xrightarrow{\text{inl} \cdot \mu} & H^V \otimes \mathbb{F}_p R
\end{array}
$$

in which $\alpha$ is the product of $1_H^V$ with the augmentation of $R$, $\beta$ is $(1_H^V) \cdot \gamma$, and $\text{inl}_i$ is the map from $H^V$ to the tensor product obtained using the unit of $R$. Since the lower horizontal map is an isomorphism, it follows that the induced map $\chi : T^V_\beta (H^V \otimes \mathbb{F}_p R) \to T^V_\alpha (H^V \otimes \mathbb{F}_p R)$ is an isomorphism. It is clear that $\chi$ respects the natural structures of its domain and range as modules over $T^V_\gamma (H^V)$, where $\gamma$ the identity map of $H^V$. The lemma follows from the fact [DW, 2.2] that the domain of $\chi$ is $T^V_\gamma (H^V) \otimes \mathbb{F}_p \gamma T^V_\gamma (R)$ while the range is $T^V_\gamma (H^V) \otimes \mathbb{F}_p T^V_\gamma (R)$.

**Lemma 4.6.** Let $V$ be an elementary abelian $p$-group, $R$ a connected object of $K$, $\gamma : R \to H^V$ a map and $\phi : R \to H^V$ the trivial map. Assume that $\epsilon : R \to T^V_\gamma (R)$ is an isomorphism. Then $\epsilon : R \to T^V_\gamma (R)$ is also an isomorphism.

**Proof:** The map $\mu$ required in 4.5 is provided by the map $R \to H^V \otimes \mathbb{F}_p T^V_\gamma (R)$ which is adjoint to the identity map of $T^V_\gamma (R)$.

**Remark 4.7:** It follows from 4.5, 4.6 and 3.2 that at least if $p = 2$ the three conditions of 4.1 are equivalent to a fourth, namely, that $QH^* X$ is locally finite as an $A_p$ module and there exists a $K$ map $H^* X \to H^V \otimes \mathbb{F}_p H^* X$ which satisfies the conditions of 4.5.
LEMMA 4.8. Let $V$ be an elementary abelian $p$-group and $\nu : S \to H^V$ the inclusion of a subalgebra over $A_p$. Then $\epsilon_\nu : S \to T^V_\nu(S)$ is an isomorphism if and only if $\nu$ includes $S$ as a Hopf subalgebra of $H^V$.

PROOF: Suppose that $\epsilon_\nu$ is an isomorphism. In this case the adjunction homomorphism $S \to H^V \otimes_{F_p} T^V_\nu(S)$ provides a map $\Delta_S : S \to H^V \otimes_{F_p} S$ which fits into a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\Delta_S} & H^V \otimes_{F_p} S \\
\nu \downarrow & & \downarrow \epsilon \otimes \nu \\
H^V & \xrightarrow{\Delta_H} & H^V \otimes_{F_p} H^V
\end{array}
$$

where $\epsilon$ is the identity map of $H^V$ and we have used the fact [L, 4.2] that $\epsilon_\nu : H^V \to H^V$ is an isomorphism. It is easy to see that $\Delta_H$ is the Hopf algebra comultiplication map on $H^V$. It now follows from the fact that the comultiplication on $H^V$ is cocommutative that $\Delta_S(S) \subset S \otimes_{F_p} S$ and thus that $S$ is a Hopf subalgebra of $H^V$.

Suppose conversely that $S$ is a Hopf subalgebra of $H^V$, and let $\phi : S \to H^V$ be the trivial map which factors through the augmentation $S \to F_p$. The Hopf algebra $H^V$ is primitively generated, and the associated restricted Lie algebra of primitives [MM, 6.7] is a free abelian restricted Lie algebra on a finite collection of generators (in dimensions 1 and 2). It follows from [MM, 6.13–6.16] that $S$ is primitively generated and is isomorphic as an algebra to a finite tensor product of exterior and polynomial algebras; in particular, $Q(S)$ is a finite unstable $A_p$ module. By the proof of (1) $\implies$ (2) in Theorem 3.2 the map $\epsilon_\phi : S \to T^V_\phi(S)$ is an isomorphism. Since the comultiplication of $S$ produces the map $\mu$ required for Lemma 4.5, an application of this lemma finishes the proof.

PROOF OF 4.1: Let $R$ denote $H^*X$, $I$ the kernel of $\gamma : R \to H^V$, $S$ the image of $\gamma$ and $\nu : S \to H^V$ the inclusion map. We will use $f$ to stand for a constant map $BV \to X$ and $\phi$ for the cohomology homomorphism induced by $f$.

(1) $\implies$ (2). The assumption that $S$ is a Hopf subalgebra of $H^V$ implies by 4.8 that $\epsilon_\nu : S \to T^V_\nu(S)$ and hence (2.3(1)) $\epsilon_\gamma : S \to T^V_\gamma(S)$ are isomorphisms. Pick $s \geq 1$ and let $M = I^s/I^{s+1}$. If we can show that $\epsilon_\nu : M \cong T^V_\nu(M)$ we will be able to finish up by imitating the proof of (1) $\implies$ (2) in Theorem 3.2. By 2.3(1) it is enough to show that $\epsilon_\nu : M \cong T^V_\nu(M)$. Proposition 2.6 ensures that $\epsilon_\gamma : H^V \otimes_{S} M \to T^V_\gamma(H^V \otimes_{S} M)$ is an isomorphism, where $\epsilon$ is the identity map of $H^V$. By 2.3(2) and [L, 4.2], however, the map $\epsilon_\gamma$ is $\epsilon \otimes_{S} \nu_{\epsilon}$, so the desired result follows from the fact that $H^V$ is free [MM, 4.4] and therefore faithfully flat as a module over $S$.

(2) $\implies$ (3). This is an immediate consequence of 2.4.

(3) $\implies$ (2). By Lemma 4.4 and Theorem 3.2 the map $\epsilon_\phi : R \to T^V_\phi(R)$ is an isomorphism. The evaluation map $m : BV \times \text{Map}(BV, X)_p \to X$ induces a cohomology homomorphism $\mu : R \to H^V \otimes_{F_p} R$ which satisfies the conditions of 4.5, so the implication follows from the conclusion of 4.5.

(2) $\implies$ (1). This implication does not in fact require the assumption that $p = 2$. The map $T^V_\gamma(R) \to T^V_\gamma(S)$ is surjective and it follows immediately from naturality that $\epsilon_\gamma : S \to T^V_\gamma(S)$ is surjective. The map $T^V_\gamma(S) \to T^V_\gamma(H^V)$ is injective, and it follows from naturality and the fact that $H^V \to T^V_\gamma(H^V)$ is injective [L, 4.2] that $S \to T^V_\gamma(S)$ is injective. By 2.3(1) the map $\epsilon_\nu : S \to T^V_\nu(S)$ is an isomorphism and hence (4.8) $S$ is a Hopf subalgebra of $H^V$.

By exactness the map $I^s \to T^V_\gamma(I^s)$ is seen to be an isomorphism if $s = 1$ and a monomorphism if $s > 1$; this first fact, though, combines with the tensor product formula (2.2) and exactness to
show that $I^s \to T^V_T(I^s)$ is an epimorphism for $s \geq 1$. Thus by exactness and 2.3(1) the maps $\epsilon_t : I^s / I^{s+1} \to T^V_T(I^s / I^{s+1})$ are isomorphisms. The proof is finished by running in reverse the argument used above at the end of the proof of $(1) \implies (2)$.

§5 Torsion in homotopy groups

In this section we will use a slight variation on the ideas of [MN] to prove Theorem 1.3.

Let $\mathbb{Z}$ denote the ring of integers, $\mathbb{Z}_p$ the additive group of $p$-adic integers, and $\mathbb{Z}/p^n$ the cyclic group of order $p^n$. The group $\mathbb{Z}/p^n$ is by definition the locally finite group obtained by taking the direct limit of the groups $\mathbb{Z}/p^n$ under the standard inclusion maps.

**Lemma 5.1.** For any finitely-generated abelian group $A$ the cohomology group $H^k(B\mathbb{Z}/p^n, A)$ is isomorphic to $\mathbb{Z}_p \otimes A$ if $k > 0$ is even and is zero if $k$ is odd. The natural map $A \to \mathbb{Z}_p \otimes A$ induces isomorphisms $H^k(B\mathbb{Z}/p^n, A) \cong H^k(B\mathbb{Z}/p^n, \mathbb{Z}_p \otimes A)$ for all $k > 0$.

**Sketch of proof:** One way to do this is to calculate the homology $H_\ast(B\mathbb{Z}/p^n, \mathbb{Z})$ as a direct limit $\lim\limits_{\rightarrow} H_\ast(B\mathbb{Z}/p^n, \mathbb{Z})$ and then pass to cohomology by using the universal coefficient theorem. The key algebraic ingredient is the fact that

$$\text{Ext}_\mathbb{Z}(\mathbb{Z}/p^n, \mathbb{Z}) \cong \text{Ext}_\mathbb{Z}(\mathbb{Z}/p^n, \mathbb{Z}_p) \cong \mathbb{Z}_p.$$ 

Let $P_nX$ stand for the $n$th Postnikov stage of the space $X$ and $k^{n+1}(X)$ for the Postnikov invariant of $X$ which lies in $H^{n+1}(P_{n-1}X, \pi_nX)$ (see [W, IX]).

**Lemma 5.2.** If $Y$ is a loop space $\Omega X$ and $Y$ has finitely-generated homotopy groups, then the Postnikov invariants of $Y$ are torsion cohomology classes.

**Proof:** This follow from [MM, p. 263]. In effect, Milnor and Moore show that the rationalized Postnikov invariants

$$k^{n+1}(Y) \otimes \mathbb{Q} \in H^{n+1}(P_{n-1}Y, \pi_n(Y) \otimes \mathbb{Q})$$

are zero. Under the stated finite generation assumption this implies that the Postnikov invariants themselves are torsion.

**Proof of 1.3:** Let $S_1$ be the set of all $k$ such that $\pi_k(X) \otimes \mathbb{Z}_p \neq 0$ and $S_2$ the set of all $k$ such that $\pi_kX$ contains $p$-torsion. The set $S_1$ is non-empty (because $H^\ast(X, \mathbb{F}_p) \neq 0$) and clearly contains $S_2$. Suppose that $S_2$ is finite. In that case we can find an integer $k$ in $S_1$ such that no integer $j$ greater than $k$ belongs to $S_2$. Let $Y = \Omega^{k-2}X$. (Note that because $X$ is 2-connected the integer $k$ is greater than 2 and $Y$ is a loop space.) By Lemma 5.1 the space $\text{Map}_\ast(B\mathbb{Z}/p^n, P_1Y)$ is contractible and hence $\text{Map}_\ast(B\mathbb{Z}/p^n, P_2Y) \cong \text{Map}_\ast(B\mathbb{Z}/p^n, K(\pi_2Y, 2))$. Because of the way in which $k$ was chosen we can thus, by Lemma 5.1 again, find an essential map $f : B\mathbb{Z}/p^n \to P_2Y$ which remains essential in the $p$-completion $(P_2Y)^\tilde{p}$. The obstructions to lifting $f$ to a map $g : B\mathbb{Z}/p^n \to Y$ are the pullbacks to $B\mathbb{Z}/p^n$ of the Postnikov invariants of $Y$ [W, p. 450]; by Lemma 5.2 these obstructions are torsion, but by Lemma 5.1 and the choice of $k$ they lie in torsion-free abelian groups. Therefore the obstructions vanish, and the lift $g$ exists. The composite $h$ of $g$ with the completion map $Y \to Y^\tilde{p}$ is non-trivial because the composite of $h$ with the projection map $Y^\tilde{p} \to P_2(Y^\tilde{p}) \cong (P_2Y)^\tilde{p}$ is essential. The adjoint of $h$ is then non-zero element of $\pi_{k-2}\text{Map}_\ast(B\mathbb{Z}/p^n, X)$, an element which by Theorem 1.2 cannot exist. This contradiction shows that $S_2$ is infinite and proves the theorem.

**References**


Dedicated to the memory of Alex Zabrodsky

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