Math 453 Abstract Algebra sample 2

Groups

1. Show that if \( f : G \to H \) is a surjective homomorphism and \( K \triangleleft G \) then \( f(K) \triangleleft H \).

2. Show that intersection \( H_1 \cap H_2 \) of two subgroups \( H_1, H_2 \leq G \). Show that if \( H_1 \triangleleft G \) then \( H_1 \cap H_2 \triangleleft H_2 \).

3. If \( r \) is a divisor of \( m \) and \( s \) is a divisor of \( n \), find a subgroup of \( \mathbb{Z}_m \oplus \mathbb{Z}_n \) that is isomorphic to \( \mathbb{Z}_r \oplus \mathbb{Z}_s \).

4. (a) Prove that \( \mathbb{R} \oplus \mathbb{R} \) under addition in each component is isomorphic to \( \mathbb{C} \).
   
   (b) Prove that \( \mathbb{R}^* \oplus \mathbb{R}^* \) under multiplication in each component is not isomorphic to \( \mathbb{C}^* \).
   
   (c) Show that there is no isomorphism from \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

5. Prove that if \( H \leq G \) and \( |G : H| = 2 \), then \( H \) is normal.

6. Let \( G = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \), \( H = \langle (2,1) \rangle \) and \( K = \langle (2,0) \rangle \). Show that \( G/H \) is not isomorphic to \( G/K \).

7. Let \( G \) be a finite group, and \( H \) be a normal subgroup of \( G \).
   
   (a) Show that the order of \( aH \) in \( G/H \) must divide the order of \( a \) in \( G \).
   
   (b) Show that it is possible that \( aH = bH \), but \( |a| \neq |b| \).

8. Suppose that \( N \triangleleft G \) and \( |G/N| = m \), show that \( x^m \in N \) for all \( x \in G \).

9. For each pair of positive integer \( m \) and \( n \), show that the map from \( \mathbb{Z} \to \mathbb{Z}_m \oplus \mathbb{Z}_n \) defined by \( x \mapsto (x \mod m, x \mod n) \) is a homomorphism. Find its kernel.

10. How many (group) homomorphisms are there from \( \mathbb{Z}_{20} \) onto \( \mathbb{Z}_8 \). How many are there to \( \mathbb{Z}_8 \)?

11. Prove that \( \phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \) by \( \phi(a, b) = a - b \) is a homomorphism. Determine the kernel.

12. (a) Let \( G \) be the group of nonzero real numbers under multiplication. Suppose \( r \) is a positive integer. Show that \( x \mapsto x^r \) is a homomorphism. Determine the kernel, and determine \( r \) so that the map is an isomorphism.
   
   (b) Let \( G \) be the group of polynomial in \( x \) with real coefficients. Define the map \( p(x) \mapsto P(x) = \int p(x) \) such that \( P(0) = 0 \). Show that \( f \) is an homomorphism, and determine its kernel.

13. (a) Determine all (group) homomorphisms from \( \mathbb{Z}_n \) to itself
   
   (b) Determine all (group) homomorphisms from \( \mathbb{Z}_{30} \) to itself with kernel \( 3\mathbb{Z}_{30} \).
Rings

14. Find all the ring homomorphisms: a) \(\mathbb{Z}_5 \rightarrow \mathbb{Z}_{10}\), b) \(\mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}\).

15. Let \(R\) be a ring.
   (a) Suppose \(a \in R\). Shown that \(S = \{x \in R : ax = xa\}\) is a subring.
   (b) Show that the center of \(R\) defined by \(Z(R) = \{x \in R : ax = xa\ \text{for all} \ a \in R\}\) is a subring.

16. Let \(R\) be a ring.
   (a) Prove that \(R\) is commutative if and only if \(a^2 - b^2 = (a + b)(a - b)\) for all \(a, b \in R\).
   (b) Prove that \(R\) is commutative if \(a^2 = a\) for all \(a \in R\).

17. Show that every nonzero element of \(\mathbb{Z}_n\) is a unit (element with multiplicative inverse) or a zero-divisor.

18. Find the characteristic of \(\mathbb{Z}_n \oplus \mathbb{Z}_m\).

19. An element \(a\) of a ring \(R\) is nilpotent if \(a^n = 0\) for some \(n \in \mathbb{N}\).
   (a) Show that if \(a\) and \(b\) are nilpotent elements of a commutative ring, then \(a + b\) is also nilpotent.
   (b) Show that a ring \(R\) has no nonzero nilpotent element if and only if 0 is the only solution of \(x^2 = 0\) in \(R\).
   (c) Show that the set of all nilpotent elements of a commutative ring is an ideal.

20. Let \(R_1\) and \(R_2\) be rings, and \(\phi : R_1 \rightarrow R_2\) be a ring homomorphism such that \(\phi(R) \neq \{0'\}\).
   (a) Show that if \(R_1\) has unity and \(R_2\) has no zero-divisors, then \(\phi(1)\) is a unity of \(R_2\).
   (b) Show that the conclusion in (a) may fail if \(R_2\) has zero-divisors.

21. Let \(R_1\) and \(R_2\) be rings, and \(\phi : R_1 \rightarrow R_2\) be a ring homomorphism.
   (a) Show that if \(A\) is an ideal of \(R_1\), then \(\phi(A)\) is an ideal of \(\phi(R_1)\).
   (b) Give an example to show that \(\phi(A)\) may not be an ideal of \(R_2\).
   (c) Show that if \(B\) is an ideal of \(R_2\), then \(\phi^{-1}(B)\) is an ideal of \(R_1\).

22. Let \(D\) be an integral domain.
   Show that a nonconstant polynomial in \(D[x]\) has no multiplicative inverse.

23. Solve the equations in \(\mathbb{Z}_7\): (a) \(x^2 = 2\), (b) \(3x = 4\)

24. Show that \(I = \{a_0 + \cdots + a_nx^n : a_i \in \mathbb{Q}, a_0 + \cdots + a_n = 0\}\) is an ideal. Show that \(A = \{a_0 + \cdots + a_nx^n : a_i \in \mathbb{Q}, a_0 + \cdots + a_n \in \mathbb{Z}\}\) is an subring of \(\mathbb{Q}[x]\).