Bridge to Research Seminar:
Applications of Singular Perturbations in Calculus of Variations

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Outline of Talks

• Examples of interfaces and defects in physical system and concept of singular perturbation
• A simple one-dimensional example illustrating selection principle, microstructure, and multiple length scales
• Connection with minimal surfaces
• Connection to point vortices


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Examples of Interfaces and Defects

**Interface**: surface separating two regions

**Defect**: localized deviations in structure from the background environment

- phase boundaries between ice and water in crystal growth
- interfaces between two immiscible fluids
- grain boundaries
- triple junctions
- soap films and soap bubbles
- dislocation lines in materials
- vortices in superconductivity
Example of Interfaces: Crystal Growth
Example of Interfaces: Bubble Growth

Motion of Defects

(Numbers indicate time in minutes after cessation of agitation.)
Example of Interfaces: Bubble Growth–2

Motion of Defects

Fig. 12—Growth and Disappearance of Bubbles in a Flat Cell
Examples of Interfaces: Grain Growth

Motion of Defects
Examples of Interfaces: Grain Growth–2

Structure of Defects
Examples of Interfaces: Grain Growth–3

Seeing the Structure and Motion of Defects with your Naked Eyes:

Figure 1. Apparatus for producing rafts of bubbles.
Example of Interfaces – Images

Image De-Noising

Fig. 3. (a) “Resolution Chart”. (b) Noisy “Resolution Chart”, SNR = 1.0. (c) Wiener filter reconstruction from (b). (d) TV reconstruction from (b).
Singular Perturbations

Higher Order Perturbations

How are the solutions of

\[ F(u) = 0 \]

in particular, the structures and locations of their singularities, if any, related to those of

\[ F(u) + \epsilon G(u, \nabla u, \nabla^2 u) = 0 \]

for \( \epsilon \ll 1 \)?

The \( \epsilon \) can be some physical parameter or even numerical discretization length scales. The key is to understand the limiting behavior as \( \epsilon \longrightarrow 0 \).
Some Well Known Examples of Singular Perturbations

Method of Vanishing Viscosity and Selection Principle

• Entropy Solution for Conservation Law

\[ U_t + F(U)_x = \epsilon^2 U_{xx} \]

converges, as \( \epsilon \to 0 \) to the entropy solution of

\[ U_t + F(U)_x = 0 \]

• Viscosity Solution for Hamilton-Jacobi Equation

\[ u_t + H(\nabla u) = \epsilon^2 \Delta u \]

converges, as \( \epsilon \to 0 \) to the viscosity solution of

\[ u_t + H(\nabla u) = 0 \]
A Common Example in Calculus of Variations

\[ \mathcal{F}(u) = \int \epsilon^2 |\nabla u|^2 + W(u) \]

or

\[ \mathcal{F}(u) = \int \epsilon^2 |\nabla^2 u|^2 + W(\nabla u) \]

where

- \( W \) is some function which is positive and vanishes on some finite set or manifold;
- \( u \) can be a scalar or vector-valued function.

The main goal is to minimize \( \mathcal{F} \) subject to some boundary conditions for \( u \).
An Example from Calculus of Variations

Consider the following minimization problem:

$$\min \left\{ \int_0^1 (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}$$
An Example from Calculus of Variations

Consider the following minimization problem:

\[
\min \left\{ \int_{0}^{1} (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]

The functional attains the \textbf{minimum value zero}. 
An Example from Calculus of Variations

Consider the following minimization problem:

\[
\min \left\{ \int_0^1 (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]

Another example of global minimizer.
An Example from Calculus of Variations

Consider the following minimization problem:

$$\min \left\{ \int_0^1 (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}$$

Yet another example of a minimizer.

Hence there are lots of examples of global minimizers, i.e. the solutions are highly non-unique!
An Example from Calculus of Variations

Consider the following *singularly perturbed* version:

\[
\min \left\{ \int_{0}^{1} \epsilon^2 u''^2 + (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]
An Example from Calculus of Variations

Consider the following **singularly perturbed** version:

\[
\min \left\{ \int_0^1 \epsilon^2 u_{xx}^2 + (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]

The minimizer have **smoothed out** corners.
An Example from Calculus of Variations

Consider the following *singly perturbed* version:

\[
\min \left\{ \int_0^1 \epsilon^2 u_{xx}^2 + (1 - u_x^2)^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]

1. each corner gives rise to a *delta function in the second derivative* which leads to *infinite* functional value;
2. the singular functional thus smoothes out the corner;
3. the amount of smoothing depends on the parameter \( \epsilon \);
4. still, each smoothed corner contributes to some energy;
5. hence *global minimizer* likes to have *as few corners* as possible.
6. thus the one with only *one corner* is the *global minimizer*. 
Variant of the Previous Example

Consider the minimization problem:

$$\min \left\{ \int_0^1 (1 - u_x^2)^2 + u^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}$$
Variant of the Previous Example

Consider the minimization problem:

$$\min \left\{ \int_0^1 (1 - u_x^2)^2 + u^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}$$

increasing number of oscillations:
the sequence of functions converge to the ZERO function which is NOT the minimizer.
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\]

increasing number of oscillations: the sequence of functions converge to the ZERO function which is NOT the minimizer.

There is a **minimizing sequence but no minimizer**! This is an example of a functional which is **not lower-semi-continuous**.
Variant of the Previous Example

Consider the following *singularly perturbed* version:

\[
\min \left\{ \int_0^1 \epsilon^2 u_{xx}^2 + (1 - u_x^2)^2 + u^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]
Variant of the Previous Example

Consider the following singularly perturbed version:

\[
\min \left\{ \int_0^1 \epsilon^2 u_{xx}^2 + (1 - u_x^2)^2 + u^2 \, dx, \quad u(0) = 0, \quad u(1) = 0 \right\}
\]

There is a balance or competition between the terms \( \int \epsilon^2 u_{xx}^2 \) and \( \int u^2 \).

small number of smoothed corners but large contribution from \( u \)

large number of smoothed corners but small contribution from \( u \)
Structure of Global Minimizer

S. Müller: the global minimizer exists and is unique and periodic with period $P = O(\epsilon^{\frac{1}{3}})$.

Note the existence of multiple (three) length scales:
- scale of the smoothed corners, defects $\epsilon$;
- scale of the pattern $P = O(\epsilon^{\frac{1}{3}})$;
- scale of the domain $O(1)$.

$\epsilon \ll \epsilon^{\frac{1}{3}} \ll O(1)$
Structure of Local Minimizer

Y.: At low energy level, all the local minimizers are periodic. As the period decreases, periodic critical points become unstable. The length scale at which this happens is also characterized. 

Energy value
Structure of Local Minimizer

Y.: At **low energy level**, all the **local minimizers** are **periodic**. As the **period decreases**, periodic critical points become **unstable**. The length scale at which this happens is also characterized. **Energy value**

The **dynamics** on the energy landscape is in fact controlled by **local minimizers**, **saddle points**, or more generally **metastable states**!
An Actual Example of Microstructure

Martensitic Transformation
An Actual Example of Microstructure

Martensitic Transformation
An Actual Example of Microstructure

Martensitic Transformation
An Example from Higher Dimension

Consider the following minimization problem:

$$\min \left\{ \mathcal{F}(u) = \int_{\Omega} (1 - u^2)^2 \, dx, \quad \int_{\Omega} u = m \right\}$$

(In the above, $u$ is a scalar-valued function. It is an example of a more general Ginzburg-Landau functional in which $u$ can be vector-valued.)
An Example from Higher Dimension

Consider the following minimization problem:

\[
\min \left\{ \mathcal{F}(u) = \int_{\Omega} (1 - u^2)^2 \, dx, \quad \int_{\Omega} u = m \right\}
\]

The minimizer is represented any subset \( A \subset \Omega \) such that

\[
\text{Area}(A) = \frac{m + \text{Area}(\Omega)}{2} : \quad u = 1 \text{ on } A \text{ and } u = -1 \text{ on } \Omega \setminus A.
\]

and hence there are infinitely many solutions. The boundary of \( A \) acts as an interface separating the regions \( \{u = 1\} \) and \( \{u = -1\} \).
An Example from Higher Dimension

Consider the following singular perturbation of $F$ (often called the Allen-Cahn Functional):

$$F_\epsilon(u) = \int_\Omega \epsilon^2 |\nabla u|^2 + (1 - u^2)^2, \quad \int_\Omega u = m$$

The term $\int_\Omega |\nabla u|^2$ penalizes rapid changes of $u$. 
An Example from Higher Dimension

The minimizer can be represented by a subset $A \subset \Omega$ such that

$$\text{Area}(A) = \frac{m + \text{Area}(\Omega)}{2}$$

but with minimum boundary length (or area) so as to minimize the contribution from $\int |\nabla u|^2$. 
An Example from Higher Dimension

The minimizer can be represented by a subset $A \subset \Omega$ such that

$$\text{Area}(A) = \frac{m + \text{Area}(\Omega)}{2}$$

but with minimum boundary length (or area)

In dimension two: the boundary will be a circular arc: curve with constant curvature which minimizes length with prescribed volume;

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An Example from Higher Dimension

The minimizer can be represented by a subset $A \subset \Omega$ such that

$$\text{Area}(A) = \frac{m + \text{Area}(\Omega)}{2}$$

but with minimum boundary length (or area)

In dimension three: the boundary will be a surface with constant mean curvature which minimizes area with prescribed volume.
Limit of the Allen-Cahn Functional

Modica-Mortola; Sternberg. \( \frac{1}{\varepsilon} \mathcal{F}_\varepsilon \longrightarrow ("\Gamma") \) \( \mathcal{F}_* \) where

\[
\mathcal{F}_* : L^1(\Omega) \longrightarrow \mathbb{R}_+,
\mathcal{F}_*(u) = \int_{\Omega} |\nabla u|
\]

for \( u : \Omega \longrightarrow \{-1, 1\} \) and \( \int_{\Omega} u = m. \) \( \mathcal{F}_*(u) = \infty \) otherwise.

Note: \( \mathcal{F}_*(u) = \mathcal{H}^{n-1}(\partial \{u = 1\}) = \text{length or area of } \partial \{u = 1\}. \)
Modeling of Point Defects – Vortices

Given $g : \partial \Omega \longrightarrow S^1$, $|g| = 1$, $\text{deg}(g, \partial \Omega) = 1$, $\Omega \subset \mathbb{R}^2$. Consider the minimization problem:

$$\min \int_{\Omega} \frac{1}{2} |\nabla u|^2, \quad u : \Omega \longrightarrow S^1, \quad u_{\partial \Omega} = g$$

A singularity – point defect must occur somewhere inside $\Omega$:
Energy of a Point Defect

Point Defect (Vortex) has infinite energy:

Near the singularity,

\[ u \sim e^{i\theta} \text{ so that } |\nabla u|^2 = |\nabla \theta|^2 \sim \frac{1}{r^2} \]

hence

\[ \int_0^1 \int_0^{2\pi} \frac{1}{r^2} r dr d\theta = 2\pi \int_0^1 \frac{1}{r} dr = \infty \]

To remedy this, need to exclude the point singularity.
Modeling of Point Defects

Given $g : \partial \Omega \longrightarrow S^1$, $|g| = 1$, $\deg(g, \partial \Omega) = 1$,
consider the minimization problem:

$$
\min \int_{\Omega_\epsilon} \frac{1}{2} |\nabla u|^2, \quad u : \Omega_\epsilon \longrightarrow S^1, \quad u_{\partial \Omega} = g, \quad \deg(u, \partial B_\epsilon(a)) = 1
$$

where $\Omega_\epsilon = \Omega \setminus B_\epsilon(a)$.
Now the energy of $u$ in $\Omega_\epsilon$ is finite:

$$\int_\epsilon^1 \int_0^{2\pi} \frac{1}{r^2} r dr d\theta = 2\pi \int_\epsilon^1 \frac{1}{r} dr = \pi \ln \frac{1}{\epsilon}$$

The limit of the minimizer, $u^\epsilon$, as $\epsilon \to 0$ is called the canonical harmonic map (as $u$ will be a harmonic function) on $\Omega \setminus \{a\}$.

(Note that the energy of $u$ will still go to $\infty$ as $\epsilon \to 0$.)
Modeling of Point Defects – Renormalized Energy

Depending on the number of defects and the degree of $u$ around each defect, the energy of $u^\epsilon$ can be shown to be:

$$\int_{\Omega_\epsilon} \frac{1}{2} |\nabla u|^2 = \pi \left( \sum_{i=1}^{N} d_i^2 \right) \ln \frac{1}{\epsilon} + O(1)$$

In order to further investigate the location of the defects, consider the next term in the expansion:

$$\int_{\Omega_\epsilon} \frac{1}{2} |\nabla u|^2 = \pi \left( \sum_{i=1}^{N} d_i^2 \right) \ln \frac{1}{\epsilon} + W(a_1, a_2, \ldots, a_N) + o(1)$$

The function $W$ can be shown to exist and is called the renormalized energy which is a function of the defect locations, $a_i$’s.
Another Example of Singular Perturbation

Let \( u : \Omega \rightarrow \mathbb{C}(\equiv \mathbb{R}^2) \).

\[
\mathcal{F}_\epsilon(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^2} (1 - |u|^2)^2, \quad u_{\partial \Omega} = g
\]

Upon minimization of \( u \), subject to appropriate Dirichlet boundary condition, it can be shown that the energy of a minimizer \( u_\epsilon \) satisfies:

\[
\mathcal{F}_\epsilon(u_\epsilon) \approx N \pi \ln \frac{1}{\epsilon} + W(a_1^*, a_2^*, \ldots, a_N^*) + N \gamma
\]

where \( N = \text{deg}(g, \partial \Omega) \) and \( \gamma \) is some universal constant. The location of the vortices \( a_i^* \)'s minimizes the renormalized energy \( W \).
Modeling of Curves (Filaments) in $\mathbb{R}^3$

Let $u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{C}(\equiv \mathbb{R}^2)$.

$$\mathcal{E}(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^2} (1 - |u|^2)^2$$
Dynamics of Curves (Filaments) in $\mathbb{R}^3$

Heat Flow (Negative Gradient Flow):

$$u_t = \Delta u + \frac{1}{\epsilon^2} u(1 - |u|^2)$$

converges to **Motion by Mean Curvature**:

$$V_N = \kappa$$  

($N$ is the normal direction).
Dynamics of Curves (Filaments) in $R^3$

Schrödinger Flow:

$$\frac{1}{i} u_t = \triangle u + \frac{1}{\epsilon^2} u (1 - |u|^2)$$

converges to bi-normal Mean Curvature Motion:

$$V_B = \kappa \quad (B = T \times N \text{ is the bi-normal direction}).$$
Thank you for your attention.