Finally, an important application for Cramer’s rule dealing with inverse matrices will be given in the next section.

### Problem Set 7.7

#### 1-6 General Problems

1. **General Properties of Determinants.** Illustrate each statement in Theorems 1 and 2 with an example of your choice.

2. **Second-Order Determinant.** Expand a general second-order determinant in four possible ways and show that the results agree.

3. **Third-Order Determinant.** Do the task indicated in Theorem 2. Also evaluate $D$ by reduction to triangular form.

4. **Expansion Numerically Impractical.** Show that the computation of an $n$th-order determinant by expansion involves $n!$ multiplications, which if a multiplication takes $10^{-9}$ sec would take these times:

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.004</td>
<td>22</td>
<td>77</td>
<td>$0.5 \cdot 10^9$</td>
</tr>
</tbody>
</table>

5. **Multiplication by Scalar.** Show that det $(kA) = k^n$ det $A$ (not $k$ det $A$). Give an example.

6. **Minors, cofactors.** Complete the list in Example 1.

#### 7-15 Evaluation of Determinants

Showing the details, evaluate:

| $\cos \alpha$ | $\sin \alpha$ | 8. | 0.4 | 4.9 |
| $\sin \beta$ | $\cos \beta$ | 1.5 | 1.3 |

9. $\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}$

10. $\begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix}$

11. $\begin{vmatrix} 4 & -1 & 8 \\ 0 & 2 & 3 \\ 0 & 5 & 0 \end{vmatrix}$

12. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

13. $\begin{vmatrix} 0 & 4 & -1 & 5 \\ -4 & 0 & 3 & -2 \\ 1 & -3 & 0 & 1 \\ -5 & 2 & -1 & 0 \end{vmatrix}$

14. $\begin{vmatrix} 0 & 4 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix}$

#### 15-19 Rank by Determinants

Find the rank by Theorem 3 (which is not very practical) and check by row reduction. Show details:

15. $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix}$

16. **CAS Experiment. Determinant of Zeros and Ones.** Find the value of the determinant of the $n \times n$ matrix $A_n$ with main diagonal entries all 0 and all others 1. Try to find a formula for this. Try to prove it by induction. Interpret $A_2$ and $A_4$ as incidence matrices (as in Problem Set 7.1 but without the minuses) of a triangle and a tetrahedron, respectively; similarly for an $n$-simplex, having $n$ vertices and $(n(n - 1)/2)$ edges (and spanning $R^{n-1}$, $n = 5, 6, \ldots$).

17. $\begin{vmatrix} 4 & 9 \\ -8 & -6 \end{vmatrix}$

18. $\begin{vmatrix} 0 & 4 & -6 \\ 16 & 12 & -6 \\ -6 & 10 & 0 \end{vmatrix}$

19. $\begin{vmatrix} 1 & 5 & 2 & 2 \\ 1 & 3 & 2 & 6 \\ 4 & 0 & 8 & 48 \end{vmatrix}$

20. **Team Project. Geometric Applications: Curves and Surfaces Through Given Points.** The idea is to get an equation from the vanishing of the determinant of a homogeneous linear system as the condition for a nontrivial solution in Cramer’s theorem. We explain the trick for obtaining such a system for the case of a line $L$ through two given points $P_1$ $(x_1, y_1)$ and $P_2$ $(x_2, y_2)$. The unknown line is $ax + by = c$, say. We write it as $ax + by + c \cdot 1 = 0$. To get a nontrivial solution $a, b, c$, the determinant of the “coefficients” $x, y, 1$ must be zero. The system is

$a_1 x + b_1 y + c_1 \cdot 1 = 0$ (Line $L$)

$a_2 x + b_2 y + c_2 \cdot 1 = 0$ (Line $L$)

$a_3 x + b_3 y + c_3 \cdot 1 = 0$ (Line $L$).
(a) Line through two points. Derive from $D = 0$ in (12) the familiar formula
\[
\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.
\]
(b) Plane. Find the analog of (12) for a plane through three given points. Apply it when the points are
\[(1, 1, 1), (3, 2, 6), (5, 0, 5)\]
(c) Circle. Find a similar formula for a circle in the plane through three given points. Find and sketch the circle through
\[(2, 6), (6, 4), (7, 1)\]
(d) Sphere. Find the analog of the formula in (c) for a sphere through four given points. Find the sphere through
\[(0, 0, 5), (4, 0, 1), (0, 4, 1), (0, 0, -3)\] by this formula or by inspection.
(e) General conic section. Find a formula for a general conic section (the vanishing of a determinant of 4th order). Try it out for a quadratic parabola and for a more general conic section of your own choice.

21–25 CRAMER’S RULE
Solve by Cramer’s rule. Check by Gauss elimination and back substitution. Show details.
\[
\begin{align*}
21. & \quad 3x - 5y = 15.5 & \quad 2x - 4y = -24 \\
& \quad 6x + 16y = 5.0 & \quad 5x + 2y = 0 \\
23. & \quad 3y - 4z = 16 & \quad 23. & \quad 3x - 2y + z = 13 \\
& \quad 2x - 5y + 7z = -27 & \quad -2x + y + 4z = 11 \\
& \quad -x - 9z = 9 & \quad x + 4y - 5z = -31 \\
25. & \quad -4w + x + y = -10 & \quad w - 4x + z = 1 \\
& \quad w - 4y + z = -7 & \quad x + y - 4z = 10
\end{align*}
\]

7.8 Inverse of a Matrix.
Gauss–Jordan Elimination

In this section we consider square matrices exclusively.
The inverse of an $n \times n$ matrix $A = [a_{jk}]$ is denoted by $A^{-1}$ and is an $n \times n$ matrix such that
\[
AA^{-1} = A^{-1}A = I
\]
where $I$ is the $n \times n$ unit matrix (see Sec. 7.2).
If $A$ has an inverse, then $A$ is called a nonsingular matrix. If $A$ has no inverse, then $A$ is called a singular matrix.
If $A$ has an inverse, the inverse is unique.
Indeed, if both $B$ and $C$ are inverses of $A$, then $AB = I$ and $CA = I$, so that we obtain the uniqueness from
\[
B = IB = (CA)B = C(AB) = CI = C.
\]
We prove next that $A$ has an inverse (is nonsingular) if and only if it has maximum possible rank $n$. The proof will also show that $Ax = b$ implies $x = A^{-1}b$ provided $A^{-1}$ exists, and will thus give a motivation for the inverse as well as a relation to linear systems. (But this will not give a good method of solving $Ax = b$ numerically because the Gauss elimination in Sec. 7.3 requires fewer computations.)

**Theorem 1**
Existence of the Inverse
The inverse $A^{-1}$ of an $n \times n$ matrix $A$ exists if and only if rank $A = n$, thus (by Theorem 3, Sec. 7.7) if and only if det $A \neq 0$. Hence $A$ is nonsingular if rank $A = n$, and is singular if rank $A < n$. 
PROOF  If A or B is singular, so are AB and BA by Theorem 3(c), and (10) reduces to 0 = 0 by Theorem 3 in Sec. 7.7.

Now let A and B be nonsingular. Then we can reduce A to a diagonal matrix \( \hat{A} = [\hat{a}_{ij}] \) by Gauss–Jordan steps. Under these operations, det A retains its value, by Theorem 1 in Sec. 7.7, (a) and (b) [not (c)] except perhaps for a sign reversal in row interchanging when pivoting. But the same operations reduce AB to \( \hat{A}B \) with the same effect on det (AB). Hence it remains to prove (10) for \( \hat{A}B \); written out,

\[
\hat{A}B = \begin{bmatrix}
\hat{a}_{11} & 0 & \cdots & 0 \\
0 & \hat{a}_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{a}_{nn}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nn}
\end{bmatrix}
= \begin{bmatrix}
\hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\
\hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn}
\end{bmatrix}
\]

We now take the determinant det (\( \hat{A}B \)). On the right we can take out a factor \( \hat{a}_{11} \) from the first row, \( \hat{a}_{22} \) from the second, \( \cdots \), \( \hat{a}_{nn} \) from the nth. But this product \( \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \) equals det \( \hat{A} \) because \( \hat{A} \) is diagonal. The remaining determinant is det B. This proves (10) for det (AB), and the proof for det (BA) follows by the same idea.

This completes our discussion of linear systems (Secs. 7.3–7.8). Section 7.9 on vector spaces and linear transformations is optional. Numeric methods are discussed in Secs. 20.1–20.4, which are independent of other sections on numerics.

PROBLEM SET 7.8

1–10 INVERSE
Find the inverse by Gauss–Jordan (or by (4b) if \( n = 2 \)).

Check by using (1).

1. \[
\begin{bmatrix}
1.80 & -2.32 \\
-0.25 & 0.60 \\
0.3 & -0.1 \\
2 & 6 \\
5 & 0
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -0.4 & 0 \\
2.5 & 0 & 0
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
5 & 4
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
cos 2\theta & sin 2\theta \\
-sin 2\theta & cos 2\theta \\
0 & 0 \\
0 & 0.1 \\
2.5 & 0 & 0
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
5 & 4 & 1
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
0 & 8 & 13 \\
0 & 3 & 5
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 8 & 0 \\
2 & 0 & 0
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 8 & 0 \\
2 & 0 & 0
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\
-\frac{3}{3} & \frac{1}{3} & \frac{3}{3} \\
\frac{4}{3} & \frac{2}{3} & -\frac{3}{3}
\end{bmatrix}
\]

11–18 SOME GENERAL FORMULAS

11. Inverse of the square. Verify \( (A^2)^{-1} = (A^{-1})^2 \) for \( A \) in Prob. 1.

12. Prove the formula in Prob. 11.
DEFINITION

Real Vector Space

A nonempty set $V$ of elements $a, b, \ldots$ is called a real vector space (or a real linear space) over the field of real numbers if $V$ is equipped with two binary operations, called vector addition and scalar multiplication, that satisfy the following axioms for all $a, b, c \in V$ and $r, s \in \mathbb{R}$:

1. **Closure under addition:** For all $a, b \in V$, the sum $a + b$ is also in $V$.
2. **Commutativity of addition:** For all $a, b \in V$, $a + b = b + a$.
3. **Associativity of addition:** For all $a, b, c \in V$, $(a + b) + c = a + (b + c)$.
4. **Identity element of addition:** There exists an element $0 \in V$ such that for all $a \in V$, $a + 0 = a$.
5. **Inverse elements of addition:** For every $a \in V$, there exists a $b \in V$ such that $a + b = 0$.
6. **Closure under scalar multiplication:** For all $a \in V$ and $r \in \mathbb{R}$, the scalar multiple $ra$ is also in $V$.
7. **Distributivity of scalar multiplication with respect to vector addition:** For all $a, b \in V$ and $r \in \mathbb{R}$, $r(a + b) = ra + rb$.
8. **Distributivity of scalar multiplication with respect to field addition:** For all $a \in V$ and $r, s \in \mathbb{R}$, $(r+s)a = ra + sa$.
9. **Associativity of scalar multiplication:** For all $a \in V$ and $r, s \in \mathbb{R}$, $r(sa) = (rs)a$.
10. **Identity element of scalar multiplication:** For all $a \in V$, $1a = a$.

We have captured the essence of vector spaces in Sec. 7.4. There, we dealt with special vector spaces called Euclidean and inner product spaces. The vector space that we are dealing with in this section will have $n$-dimensional vectors as elements, with $n$ real numbers as components.
PROBLEM SET 7.9

1. **Basis.** Find three bases of \( R^2 \).

2. **Uniqueness.** Show that the representation \( \mathbf{v} = c_1 \mathbf{a}_1 + \cdots + c_n \mathbf{a}_n \) of any given vector in an \( n \)-dimensional vector space \( V \) in terms of a given basis \( \mathbf{a}_1, \cdots, \mathbf{a}_n \) for \( V \) is unique. **Hint.** Take two representations and consider the difference.

### 3–10 VECTOR SPACE

(More problems in Problem Set 9.4.) Is the given set, taken with the usual addition and scalar multiplication, a vector space? Give reason. If your answer is yes, find the dimension and a basis.

- **3**. All vectors in \( R^3 \) satisfying \(-v_1 + 2v_2 + 3v_3 = 0, -4v_1 + v_2 + v_3 = 0\).
- **4**. All skew-symmetric \( 3 \times 3 \) matrices.
- **5**. All polynomials in \( x \) of degree 4 or less with nonnegative coefficients.
- **6**. All functions \( y(x) = a \cos 2x + b \sin 2x \) with arbitrary constants \( a \) and \( b \).
- **7**. All functions \( y(x) = (ax + b)e^{-x} \) with any constant \( a \) and \( b \).
- **8**. All \( n \times n \) matrices \( \mathbf{A} \) with fixed \( n \) and \( \det \mathbf{A} = 0 \).
- **9**. All \( 2 \times 2 \) matrices \( [a_{ij}] \) with \( a_{11} + a_{22} = 0 \).
- **10**. All \( 3 \times 2 \) matrices \( [a_{ij}] \) with first column any multiple of \( [3,0,-5]^T \).

### 11–14 LINEAR TRANSFORMATIONS

Find the inverse transformation. Show the details.

**11**. \( y_1 = 0.5x_1 - 0.5x_2 \) \( y_2 = 1.5x_1 - 2.5x_2 \)

**12**. \( y_1 = 3x_1 + 2x_2 \) \( y_2 = 4x_1 + x_2 \)

**13**. \( y_1 = 5x_1 + 3x_2 - 3x_3 \)

**14**. \( y_1 = 0.2x_1 - 0.1x_2 \)

**15–20** **EUCLIDEAN NORM**

Find the Euclidean norm of the vectors:

**15**. \( [3,1,-4]^T \)

**16**. \( \left[ \frac{1}{2}, \frac{1}{3}, -\frac{1}{2}, -\frac{3}{2} \right]^T \)

**17**. \( \left[ 1, 0, 0, 1, -1, 0, -1, 1 \right]^T \)

**18**. \( (-4, 8, -1)^T \)

**19**. \( \left[ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, 0 \right]^T \)

**20**. \( \left[ \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right]^T \)

### 21–25 INNER PRODUCT. ORTHOGONALITY

**21**. Orthogonality. For what value(s) of \( k \) are the vectors \( [2, 0, 0, -4]^T \) and \( [5, k, 0, 0, \frac{1}{4}]^T \) orthogonal?

**22**. Orthogonality. Find all vectors in \( R^3 \) orthogonal to \( [2, 0, 1]^T \). Do they form a vector space?

**23**. Triangle inequality. Verify (4) for the vectors in Probs. 15 and 18.

**24**. Cauchy–Schwarz inequality. Verify (5) for the vectors in Probs. 16 and 19.

**25**. Parallelogram equality. Verify (5) for the first two column vectors of the coefficient matrix in Prob. 13.

### CHAPTER 7 REVIEW QUESTIONS AND PROBLEMS

1. What properties of matrix multiplication differ from those of the multiplication of numbers?

2. Let \( A \) be a \( 100 \times 100 \) matrix and \( B \) a \( 100 \times 50 \) matrix. Are the following expressions defined or not? \( A + B, A^2, B^2, AB, BA, AA^T, B^TA, BA^T, BB^T, B^TAB \). Give reasons.

3. Are there any linear systems without solutions? With one solution? With more than one solution? Give simple examples.

4. Let \( C \) be \( 10 \times 10 \) matrix and \( a \) a column vector with 10 components. Are the following expressions defined or not? \( Ca, C^T a, Ca^T, aC, a^T C, (Ca)^T \).

5. Motivate the definition of matrix multiplication.

6. Explain the use of matrices in linear transformations.

7. How can you give the rank of a matrix in terms of row vectors? Of column vectors? Of determinants?

8. What is the role of rank in connection with solving linear systems?

9. What is the idea of Gauss elimination and back substitution?

10. What is the inverse of a matrix? When does it exist? How would you determine it?