Exam III Solution

Letter Grade Estimation
This is for reference only.
Your course grade will be based on homework + quiz + four exams

Average: 67.8

A ≥ 91
B ≥ 74
C ≥ 53
D ≥ 37
1. (8 pt) Given 
\[ f(x) = x^4, \]
(1) find its critical point \((c, f(c))\) and use the First Derivative Test for Relative Extrema to decide whether it is a relative maximum, relative minimum, or neither.

**Solution.** From \(f'(x) = 0\), i.e., \(4x^3 = 0\), we have \(x = 0\). Thus \((0, f(0)) = (0, 0)\) is the only critical point. Furthermore, this point divides the number line into two intervals described by \(x < 0\) and \(x > 0\), respectively. Since \(f'(x) < 0\) when \(x < 0\) and \(f'(x) > 0\) when \(x > 0\), by the First Derivative Test for Relative Extrema, \((0, 0)\) is a relative minimum.

| The critical point of \(f(x)\) is \((0, 0)\) |
| Is it a relative maximum, a relative minimum, or neither? Relative minimum |

(2) Can we use the Second Derivative Test for Relative Extrema instead to classify the above critical point?

**Solution.** We have \(f''(x) = 12x^2\). So at the critical number \(x = 0\), \(f''(x) = f''(0) = 0\). Hence, the Second Derivative Test for Relative Extrema is inconclusive.

| Yes/No? and Explain. No because \(f''(x) = 0\). |
2. (12 pt) Find the horizontal and vertical asymptotes of the given function

\[ f(x) = \begin{cases} 
  \frac{x^2 + 4x + 3}{x^2 + 5x + 6} & \text{if } x < 0 \\
  \frac{-x + 2}{x - 4} & \text{if } x \geq 0 
\end{cases} \]

**Solution.** To find the horizontal asymptotes, we take the limits when \( x \to \infty \) and \( x \to -\infty \), respectively.

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{-x + 2}{x - 4} = \lim_{x \to \infty} \frac{-1 - \frac{2}{x}}{1 - \frac{4}{x}} = -1.
\]

So \( y = -1 \) is a horizontal asymptote.

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 + 4x + 3}{x^2 + 5x + 6} = \lim_{x \to -\infty} \frac{1 + \frac{4}{x} + \frac{3}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}} = 1.
\]

So \( y = 1 \) is also a horizontal asymptote.

Since when \( x > 0 \), \( f(x) = \frac{-x + 2}{x - 4} \), where the numerator and denominator have no common factors, \( x = 4 \) is a vertical asymptote.

When \( x < 0 \), \( f(x) = \frac{x^2 + 4x + 3}{x^2 + 5x + 6} = \frac{(x+1)(x+3)}{(x+2)(x+3)} = \frac{x+1}{x+2} \) where \( x \neq -3 \). Therefore, \( x = -2 \) is also a vertical asymptote.

The horizontal asymptote(s) is(are) \( y = 1 \) and \( y = -1 \)

The vertical asymptote(s) is(are) \( x = 4 \) and \( x = -2 \)
3. (20 pt) Sketch the graph of the given function after filling in the blanks. [If you find nothing to fill in a blank, just put NONE.]

\[ f(x) = x^3 - 3x^2 \]

**Solution.** We immediately have \( f(x) = x^2(x-3) \), \( f'(x) = 3x(x-2) \) and \( f''(x) = 6(x-1) \). Then

The domain of \( f(x) \): all real numbers

Points of intersection \((c, f(c))\) with the x or y axes: \((0, 0), (3, 0)\)

Horizontal asymptotes: None

Vertical asymptotes: None

Intervals of increase: \( x < 0, x > 2 \)

Intervals of decrease: \( 0 < x < 2 \)

Local maxima \((c, f(c))\): \((0, 0)\)

Local minima \((c, f(c))\): \((2, -4)\)

Intervals of concave upward: \( x > 1 \)

Intervals of concave downward: \( x < 1 \)

Inflection points \((c, f(c))\): \((1, -2)\)

Sketch the graph below:

Remark: The same function showed up in Quiz #11.
4. (20 pt) Sketch the graph of the given function after filling in the blanks. [If you find
nothing to fill in a blank, just put NONE.]

\[ f(x) = \frac{x^2 - 9}{x^2 + 1} \]

(You can use the facts \( f'(x) = \frac{20x}{(x^2 + 1)^2} \) and \( f''(x) = \frac{-60(x + \sqrt{3})(x - \sqrt{3})}{(x^2 + 1)^3} \).)

The domain of \( f(x) \): all real numbers

Points of intersection \((c, f(c))\) with the x or y axes: \((-3, 0), (3, 0), (0, -9)\)

Horizontal asymptotes: \( y = 1 \)

Vertical asymptotes: None

Intervals of increase: \( x > 0 \)

Intervals of decrease: \( x < 0 \)

Local maxima \((c, f(c))\): None

Local minima \((c, f(c))\): \((0, -9)\)

Intervals of concave upward: \( -\frac{\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3} \)

Intervals of concave downward: \( x < -\frac{\sqrt{3}}{3}, x > \frac{\sqrt{3}}{3} \)

Inflection points \((c, f(c))\): \( (-\frac{\sqrt{3}}{3}, -\frac{13}{2}), (\frac{\sqrt{3}}{3}, -\frac{13}{2}) \)

Sketch the graph below:

Remark: This is the last problem in Quiz #9
5. (10 pt) Use the Extremum Value Property to find the absolute maximum of

\[ f(x) = 3x^4 - 4x^3 \]

on the interval given by \(-1 \leq x \leq 2\).

**Solution.** The Extremum Value Property says a continous function on a closed interval must have an absolute maximum and an absolute minimum and they either occur at critical numbers or at the end points of the interval.

So to find the absolute maximum, we start to look at

\[ f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1). \]

So the critical numbers are \(x = 0\) and \(x = 1\), which are in the interval \(-1 \leq x \leq 2\). The end points of the interval \(-1 \leq x \leq 2\) are \(x = -1\) and \(x = 2\).

Then we have \(f(-1) = 7\), \(f(0) = 0\), \(f(1) = -1\) and \(f(2) = 16\). So \(f(x)\) attains its absolute maximum value 16 at \(x = 2\).

The absolute maximum \((c, f(c))\) is \((2, 16)\)
6. (10 pt) An open box is to be made from a square piece of cardboard, 6 inches by 6 inches, by removing a small square with side length $x$ from each corner (see the figure) and folding up the flaps to form the sides. Use the **Second Derivative Test for Absolute Extrema** to find the $x$ value maximizing the volume of the box.

**Solution.** As seen in the figure, the length of the sides of the base square is $6 - 2x$, which must be greater than 0, i.e., $6 - 2x > 0$. So $x < 3$. On the other hand, the height of the box $x$ is also greater than 0. So $x > 0$. Putting them together, we have $0 < x < 3$.

The volume $V$ now can be written as a function of $x$:

$$V(x) = x(6 - 2x)^2,$$

whose domain is described by the previously found inequalities $0 < x < 3$, which is an (open) interval.

Rewriting $V(x)$ as $4x(3 - x)^2 = 4x(9 - 6x + x^2) = 4(9x - 6x^2 + x^3)$, we immediately have $V'(x) = 4(9 - 12x + 3x^2) = 12(3 - 4x + x^2) = 12(x - 1)(x - 3)$. Thus, the solution to $V'(x) = 0$ inside the interval $0 < x < 3$ is $x = 1$. This means $x = 1$ is the only critical number of $V(x)$ over its interval domain.

Furthermore, we find that $V''(x) = 12(-4 + 2x)$. So $V''(1) = 12(-4 + 2) < 0$. Therefore, by the **Second Derivative Test for Absolute Extrema**, $x = 1$ maximizes the volume.
7. (10 pt)

(1) Given

\[ 3^{x^2} \left( \frac{1}{9} \right)^x = 27, \]

solve for \( x \).

**Solution.** Writing \( \frac{1}{9} \) as \( 3^{-2} \) and 27 as \( 3^3 \), we have

\[ 3^{x^2} (3^{-2})^x = 3^3, \text{ i.e., } 3^{x^2} 3^{-2x} = 3^3. \]

Then we have \( 3^{x^2-2x} = 3^3 \).

By the equality rule for exponential functions, \( x^2 - 2x = 3 \).

Thus, \( x^2 - 2x - 3 = 0 \).

After factorization, \( (x - 3)(x + 1) = 0 \).

So \( x = 3 \) or \( x = -1 \).

\[ x = 3, \text{ or } x = -1 \]

**Remark:** This is the same as Problem 2 of Quiz #12.

(2) An investment of $100 grows to $110 in 2 years. What is the annual interest rate \( r \) if the interest is compounded semi-annually?

**Solution.** We have \( 110 = 100(1 + \frac{r}{2})^{2 \cdot 2} \). Thus

\[ r = 2(1.1^{\frac{3}{2}} - 1) \approx 0.04823 = 4.823\% \]

\[ r = 4.823\% \]
8. (10 pt)

(1) Given \( f(x) = x^2 e^{\frac{1}{x}} \), find \( f'(x) \).

Solution. By the chain rule,

\[
f'(x) = (x^2)' e^{\frac{1}{x}} + x^2 (e^{\frac{1}{x}})'
= 2xe^{\frac{1}{x}} + x^2 e^{\frac{1}{x}} \left(-\frac{1}{x^2}\right)
= e^{\frac{1}{x}} (2x - 1)
\]

\[f'(x) = e^{\frac{1}{x}} (2x - 1)\]

(2) Given \( g(x) = \ln \left[ \frac{(x + 1)^5 (x + 2)^6}{(x - 2)^4 (x - 1)^8} \right] \), find \( g'(0) \).

Solution. By the product rule, quotient rule and power rule of logarithmic functions,

\( g(x) = 5 \ln(x + 1) + 6 \ln(x + 2) - 4 \ln(x - 2) - 8 \ln(x - 1) \). Therefore,

\[
g'(x) = \frac{5}{x + 1} + \frac{6}{x + 2} - \frac{4}{x - 2} - \frac{8}{x - 1}
\]

So

\[g'(0) = 5 + 3 + 2 + 8 = 18\]

\[g'(0) = 18\]