The Radon-Nikodym Theorem

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**Theorem 1** (Johann Radon-Otton Nikodym). Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space and let \(\nu\) be a measure defined on \(\mathcal{B}\) such that \(\nu \ll \mu\). Then there is a unique nonnegative measurable function \(f\) up to sets of \(\mu\)-measure zero such that

\[
\nu(E) = \int_E f \, d\mu,
\]

for every \(E \in \mathcal{B}\). \(f\) is called the Radon-Nikodym derivative of \(\nu\) with respect to \(\mu\) and it is often denoted by \(\left(\frac{d\nu}{d\mu}\right)\).

The assumption that the measure \(\mu\) is \(\sigma\)-finite is crucial in the theorem (but we don’t have this requirement directly for \(\nu\)). Consider the following two examples in which the \(\mu\)'s are not \(\sigma\)-finite.

**Example 1.** Let \((\mathbb{R}, \mathcal{M}, \nu)\) be the Lebesgue measure space. Let \(\mu\) be the counting measure on \(\mathcal{M}\). So \(\mu\) is not \(\sigma\)-finite. For any \(E \in \mathcal{M}\), if \(\mu(E) = 0\), then \(E = \emptyset\) and hence \(\nu(E) = 0\). This shows \(\nu \not\ll \mu\). Do we have the associated Radon-Nikodym derivative? Never. Suppose we have it and call it \(f\), then for each \(x \in \mathbb{R}\), \(0 = \nu(\{x\}) = \int_{\{x\}} f \, d\mu = \int_{\{x\}} f \chi_{\{x\}} \, d\mu = f(x)\mu(\{x\}) = f(x)\). Hence, \(f = 0\).

This means for every \(E \in \mathcal{M}\), \(\nu(E) = \int_E 0 \, d\mu = 0\), which contradicts that \(\nu\) is the Lebesgue measure.

**Example 2.** \(\nu\) is still the same measure above but we let \(\mu\) to be defined by \(\mu(\emptyset) = 0\) and \(\mu(A) = \infty\) if \(A \neq \emptyset\). Clearly, \(\mu\) is not \(\sigma\)-finite and \(\nu \ll \mu\). The Radon-Nikodym derivative does not exist, neither. Suppose \(f\) is one. Then for any \(x \in \mathbb{R}\), \(0 = \nu(\{x\}) = \int_{\{x\}} f \, d\mu = f(x)\mu(\{x\}) = f(x)\infty\). Thus, \(f = 0\).

Hence for any \(E \in \mathcal{M}\), \(\nu(E) = \inf_E 0 \, d\mu = 0\).

[Also notice that if \(\mu(X) = 0\), then the measure of every set in \(\mathcal{M}\) with respect to \(\mu\) and \(\nu\) is zero, which is not interesting at all. So we ignore this case.]

Now we prove the Radon-Nikodym Theorem.

*Proof (Wikipedia):* We assume that \((X, \mathcal{B}, \mu)\) is a finite measure space and show the existence and uniqueness of the Radon-Nikodym derivative. Then the case when \(\mu\) is \(\sigma\)-finite follows by the “paste- ing” of the derivatives on each set of finite measure.

Let \(\mathcal{F} = \{f \text{ is measurable} | \nu(E) \geq \int_E f \, d\mu, \text{ for all } E \in \mathcal{M}\}\). \(\mathcal{F}\) is nonempty because the zero function is in it. Let \(s = \sup_{f \in \mathcal{F}} \int_X f \, d\mu\). Then there is a sequence \(\{h_n\}\) in \(\mathcal{F}\) such that \(\lim_{n \to \infty} \int_X f_n \, d\mu = s\).

Let \(f_1, f_2 \in \mathcal{F}\), then for any \(E \in \mathcal{M}\), \(\int_E f_1 \vee f_2 \, d\mu = \int_{\{x \in E | f_1(x) \geq f_2(x)\}} f_1 \vee f_2 \, d\mu + \int_{\{x \in E | f_1(x) < f_2(x)\}} f_1 \vee f_2 \, d\mu = \int_{\{x \in E | f_1(x) \geq f_2(x)\}} f_1 \, d\mu + \int_{\{x \in E | f_1(x) < f_2(x)\}} f_2 \, d\mu \leq \nu(\{x \in E | f_1(x) \geq f_2(x)\}) + \nu(\{x \in E | f_1(x) < f_2(x)\}) = \nu(E)\). Therefore, \(f_1 \vee f_2 \in \mathcal{F}\).

Let \(f_n = \bigvee_{k=1}^n h_k\). Then \(\{f_n\}\) is a nonnegative increasing sequence in \(\mathcal{F}\) and \(\lim_{n \to \infty} \int_X f_n \, d\mu = s\).

Define \(g\) by \(g(x) = \lim_{n \to \infty} f_n(x)\) for \(x \in X\). Then by the monotone convergence theorem, for any \(E \in \mathcal{M}\), \(\int_E g \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu \leq \nu(E)\). This shows \(g \in \mathcal{F}\) and \(\int_X g \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu = s\).
Therefore, the function $\nu_0$ defined on $\mathcal{M}$ by $\nu_0(E) = \nu(E) - \int_E g d\mu$ is a measure. We want to show that $\nu_0 = 0$ and then $g$ is the desired function. Suppose $\nu_0$ is not zero. Since $\nu_0(X) > 0$ and $\mu(X) < \infty$, there is $\epsilon > 0$ such that $\nu_0(X) - \epsilon\mu(X) > 0$. Let $\{A, B\}$ be a Hahn decomposition for the signed measure $\nu_0 - \epsilon\mu$. Then for every $E \in \mathcal{M}$, $\nu_0(A \cap E) - \epsilon\mu(A \cap E) \geq 0$. So, $\nu(E) = \nu_0(E) + \int_E g d\mu \geq \nu_0(E \cap A) + \int_E g d\mu \geq \epsilon\mu(A \cap E) + \int_E g d\mu = \int_E (g + \epsilon\chi_A)$. Therefore, $g + \epsilon\chi_A$ is also in $\mathcal{F}$. However, if $\mu(A) > 0$, then $\int_X (g + \epsilon\chi_A) d\mu = \int_X g d\mu + \epsilon\mu(A) > \int_X g d\mu = s$, which is a contradiction. In fact, if $\mu(A) = 0$, since $\nu \ll \mu$, $\nu(A) = 0$. So $\nu_0(A) = \nu(A) - \int_A g d\mu \leq \nu(A) = 0$. Hence $\nu_0(A) = 0$. Consequently, $\nu_0(X) - \epsilon\mu(X) = \nu_0(B) - \epsilon\mu(B) \leq 0$, contradicting that $\nu_0(X) - \epsilon\mu(X) > 0$.

Therefore, $\nu_0 = 0$, which means $\nu(E) = \int_E g \mu$ for every $E \in \mathcal{M}$.

To show uniqueness, let $\nu(E) = \int_E f d\mu = \int_E g d\mu$. Then $\int_E (f - g) d\mu = 0$. Since $E$ is arbitrary, $\int_{\{f > g\}} (f - g) d\mu = 0$. This shows $f = g$ a.e. on $\{x \in X | f(x) \geq g(x)\}$. Similarly, $f = g$ a.e. on $\{x \in X | f(x) < g(x)\}$. Hence $f = g$ a.e. on $X$. □

Remark: If we add the condition that $\nu$ is finite, then the function $g$ in our proof is integrable.