Riesz-Fischer Theorem

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**Theorem 1.** \( L^p(X) \) (1 ≤ p < ∞) is complete.

**Proof.** Let \( \{f_n\} \) be a Cauchy sequence in \( L^p(X) \) (1 ≤ p < ∞). Then for any \( \epsilon > 0 \), there is \( N(\epsilon) \) such that if \( m, n \geq N(\epsilon) \), then \( \|f_m - f_n\|_p < \epsilon \). Let \( n_k = N(\frac{1}{2^k}) \). Then the subsequence \( \{f_{n_k}\} \) satisfies \( \|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2^k} \).

Define the function \( f \) by

\[ f(x) = f_n + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k}), \quad x \in X. \]

Notice that the partial sum \( S_N := f_n + \sum_{k=1}^{N-1} (f_{n_{k+1}} - f_{n_k}) \) is just \( f_{n_N} \). Also define the function \( g \) by

\[ g(x) = |f_n| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|, \quad x \in X. \]

Let the partial sum of \( g \) be \( S_N(g) := |f_n| + \sum_{k=1}^{N-1} |f_{n_{k+1}} - f_{n_k}| \). Then by Minkowski’s inequality, \( \|S_N(g)\|_p \leq \|f_n\|_p + \left\| \sum_{k=1}^{N-1} |f_{n_{k+1}} - f_{n_k}| \right\|_p \leq \|f_n\|_p + \sum_{k=1}^{N-1} \sum_{k=1}^{N-1} \|f_{n_{k+1}} - f_{n_k}\|_p < \|f_n\|_p + \sum_{k=1}^{N-1} \frac{1}{2^k} \).

So the increasing sequence \( \left\{ \|S_N(g)\|_p \right\} \) is bounded above by \( \|f_n\|_p + 1 \), which shows \( \int_X g^p < \infty \). Obviously, \( |f| \leq g \). Consequently, \( \int_X |f|^p \leq \int_X g^p \). Hence, \( f = f^+ - f^- \in L^p(X) \) and it follows that \( f^p \) is integrable. Hence, the series \( -\infty < f < \infty \) a.e., or \( f \) converges a.e. on \( X \). Thus, \( \{f_{n_k}\} \) converges to \( f \) a.e..

Notice that \( |f - f_{n_N}| = |S_\infty(f) - S_{N-1}(f)| = \left| \sum_{k=N}^{\infty} (f_{n_{k+1}} - f_{n_k}) \right| \leq g \). So by the Lebesgue dominated convergence theorem, \( \lim_{k \to \infty} \|f - f_{n_k}\|_p = \left( \int_X \lim_{k \to \infty} (f(x) - f_{n_k}(x))^p dx \right)^{\frac{1}{p}} = \left( \int_X 0 \right)^{\frac{1}{p}} = 0 \). This shows the subsequence \( \{f_{n_k}\} \) converges to \( f \) in \( L^p(X) \). But \( \{f_n\} \) is itself Cauchy. This means \( f_n \) converges to \( f \) in \( L^p(X) \). \( \square \)