MODELING AND HOVER CONTROL
OF A DOUBLE-ROTOR MICRO FLYING ROBOT
VIA SHAPE CHANGE

A Thesis
Submitted to the Faculty
of
Purdue University
by
Yongheng Zhang

In Partial Fulfillment of the
Requirements for the Degree
of
Master of Science in Engineering

August 2009
Purdue University
Hammond, Indiana
“But as long as one steps in a firm, clear direction, with long strides and sharp vision, one would need far, far less than the millions of steps needed to journey a thousand miles.”

The 15 years old Terence Tao
ACKNOWLEDGMENTS

Nasser Houshangi is the first academic advisor in my life. He taught me how to do research since I was an undergraduate student. It has been a privilege for me to grow under his insightful guidance and invaluable advice, and with the trust and freedom he gave for me to choose the thesis topic myself. This thesis would not have been possible without his constant encouragement.

I would like to thank Nicolae Tarfulia and Jianghai Hu for their willingness to serve in the thesis committee.

I want to thank Stanislaw Zák for his suggestion at the beginning stage of the thesis.

I would like to thank Sangbum Cho, on whose Ph.D. dissertation the second half of my thesis is based, for the feedback and encouragement from him.

I would also like to thank my friend Zoltán Szekely for the conversation on the algebraic loop in my first attempt to build the second model in Simulink.

I also want to thank Mark Senn for the ease of writing of my thesis in \LaTeX.

In no ways the work of this thesis can be seperated from my three years’ life at Purdue University Calumet where I am indebted to many people. First, I am deeply grateful to Chenn Q Zhou. I am lucky to be one of the first passengers on the ETIE program’s vehicle she led initiating in 2006. I want to thank my advisor for his courses on parameter identification and robotics. I would like to thank Toma Hentea for his courses on discrete events simulation and neural networks. I would like to thank Howard Gerber, William Umlauf (ArcelorMittal), and Dianbing Huang (ArcelorMittal) for their guidance on my senior design project. I also want to thank Yeow Siow for introducing me to fluid mechanics and teaching me how to stimulate freshmen’s interests in engineering as a mentor. Working for Masoud Mojtahed as a
teaching assistant in the kinematics analysis and design class and the manufacturing lab has been an enjoyable and rewarding experience.

The online engineering courses from the West Lafayette campus also contribute to my thesis and constitute an integral part of my education. Many thanks to Anastasios S. Lyrintzis for motivating the helicopter aerodynamics part in my thesis. Special thanks to James M. Longuski for laying the fundations of the modeling of multi rigid body system for my thesis. I am grateful to Bin Yao for his patience in explaining to me some points in control system design. I would also like to thank James S. Bethel for teaching me digital photogrammetry. Sitting in Eric A. Nauman’s Human Motion Kinetics class was a great fun.

I owe a great deal to Weihua Ruan at the Math Department who brought my mathematics education from the “pre-rigorous” stage to the “rigorous” stage. I was exposed to the fascinating realm of mathematics in my first independent study class with him. I also owe a tremendous debt of gratitude to algebraic topologist Anthony D. Elmendorf for instilling the passion of pure mathematics and mentoring me meticulously two hours every week. I frequently received his comments on my proof after two o’clock at midnight which shaped my view of how to be professional. I also want to thank John J. Coffey for the three courses on analysis.

David Detmer, José Castro-Urioste and Geoffrey R. Barrow have made my two summers’ stay here so much delightful. Special thanks to them for letting me breathing a fresh air of philosophy, art and language aside from engineering and mathematics.

I extend my gratitude to the Skills Assessment and Development Center where I have been a supplemental instructor and tutor since 2008.

I also want to thank all my friends in the engineering and mathematics department, who shaped my life and from whom I learned a lot.

Lastly, I want to thank my beloved families in China. Their unconditional love for me made it possible to complete the greatest family project ever three years ago. Therefore, I was able to come here. I dedicate this thesis to them.
TABLE OF CONTENTS

LIST OF TABLES ......................................................... vii
LIST OF FIGURES ....................................................... viii
ABSTRACT ................................................................. ix
1 Introduction .......................................................... 1
2 Conceptual Design of C-fly ........................................ 3
3 Modeling of C-fly As a Quasi Rigid Body ....................... 6
   3.1 Rotational Dynamics ........................................... 8
   3.2 Translational Dynamics ....................................... 10
   3.3 Rotors and Mass Block Dynamics ............................ 11
   3.4 State Space Model ............................................. 11
4 Attitude Stabilization Based on the First Model ............... 14
   4.1 Control System Design ........................................ 14
   4.2 Simulation Results ............................................. 16
5 Modeling of C-fly As a Multi Rigid-Body System ............... 22
   5.1 Hamilton’s Principle ........................................... 22
   5.2 The Lagrangian ................................................ 23
   5.3 Equations of Motion .......................................... 28
   5.4 State Space Equations ........................................ 34
   5.5 Simulation ...................................................... 36
      5.5.1 Equilibrium of the System .............................. 37
      5.5.2 Numerical Simulation of the System at the Equilibrium .. 37
      5.5.3 Simulation of the System When the Mass Block is Perturbed Slightly ......................................................... 38
6 Attitude Stabilization Based on the Second Model ............. 40
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 Conclusion</td>
<td>47</td>
</tr>
<tr>
<td>LIST OF REFERENCES</td>
<td>49</td>
</tr>
<tr>
<td>APPENDIX</td>
<td></td>
</tr>
<tr>
<td>A “Analytical Experiment” of a Simple Multi Rigid Body System</td>
<td>50</td>
</tr>
</tbody>
</table>
**LIST OF TABLES**

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Simulation Parameters for C-fly</td>
<td>17</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 The upper and lower rotors from different perspectives</td>
<td>3</td>
</tr>
<tr>
<td>2.2 Fuselage of C-fly</td>
<td>4</td>
</tr>
<tr>
<td>2.3 The CoG shifting mechanism</td>
<td>4</td>
</tr>
<tr>
<td>2.4 Illustration of C-fly</td>
<td>5</td>
</tr>
<tr>
<td>3.1 Reference (OXYZ) and local (oxyz) coordinate frames and the four components of C-fly: 1. upper rotor, 2. lower rotor, 3. fuselage, and 4. mass block.</td>
<td>6</td>
</tr>
<tr>
<td>3.2 Location of the center of mass of C-fly. (a) Anerior View. (b) Lateral View on the right. (c) Inferior View.</td>
<td>8</td>
</tr>
<tr>
<td>4.1 Rotor speeds and mass block position</td>
<td>18</td>
</tr>
<tr>
<td>4.2 Rotational motion</td>
<td>19</td>
</tr>
<tr>
<td>4.3 Rotor speeds and mass block position</td>
<td>20</td>
</tr>
<tr>
<td>4.4 Rotational motion</td>
<td>21</td>
</tr>
<tr>
<td>5.1 System Response at the Equilibrium Point</td>
<td>38</td>
</tr>
<tr>
<td>5.2 System Response When the Mass Block is Perturbed Slightly</td>
<td>39</td>
</tr>
<tr>
<td>6.1 System response when ((\phi_0, \theta_0, \psi_0) = (17.2^\circ, 17.2^\circ, 17.2^\circ))</td>
<td>41</td>
</tr>
<tr>
<td>6.2 System response when ((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ), a_0 = 0.1\text{cm}) and (b_0 = 0.1\text{cm})</td>
<td>42</td>
</tr>
<tr>
<td>6.3 System response when ((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)) and the initial value of ((\omega_x, \omega_y, \omega_z) = (1\text{rad/s}, 1\text{rad/s}, 1\text{rad/s}))</td>
<td>43</td>
</tr>
<tr>
<td>6.4 System response when ((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ), \Omega_u = 150\text{rad/s}) and (\Omega_l = 300\text{rad/s})</td>
<td>44</td>
</tr>
<tr>
<td>6.5 System response when ((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)) and the measurements of the three angles are corrupted by zero mean band-limited Gaussian noises with correlation time 0.01 second and covariance 0.005 rad</td>
<td>45</td>
</tr>
<tr>
<td>6.6 System response when ((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)) and the measurements of the three angles are corrupted by zero mean band-limited Gaussian noises with correlation time 0.01 second and covariance 0.005 rad.</td>
<td>45</td>
</tr>
<tr>
<td>A.1 A schematic drawing of a simple multibody system</td>
<td>52</td>
</tr>
</tbody>
</table>
ABSTRACT

Zhang, Yongheng M.S.E., Purdue University, August 2009. Modeling and Hover Control of a Double-Rotor Micro Flying Robot via Shape Change. Major Professor: Nasser Houshangi.

Research in Micro Aerial Vehicles (MAVs) has been proliferating in both academia and industry during recent years due to their potential applications, e.g. search and rescue in the aftermath of an earthquake. Among others, coaxial double rotor MAVs have the desirable features in restricted indoor flight. However, the traditional mechanism for controlling rotorcraft’s attitude is too complex to apply on the MAVs. In this research, a conceptual design of a coaxial double-rotor micro flying robot which controls its attitude by the Center of Gravity (CoG) shifting mechanism is presented. A quasi rigid body model and a multi rigid body system model are derived respectively for the robot which is the main contribution of this research. Linear Quadratic Regulator (LQR) based on the two models which stabilize the attitude are also studied. Simulation results show the feasibility of the approaches.
1. INTRODUCTION

Research in Unmanned Aerial Vehicles (UAVs), especially their small-version Micro Aerial Vehicles (MAVs) has been proliferating in both academia and industry during recent years. Despite human’s original desire to overcome their biological limits by building flying vehicles which can move in three dimensional space [1], this activity is mainly driven by MAVs’ potential applications. In October 2007, protesters in Washington D.C. claimed that they saw insect-like vehicles hovering in the sky [2]. Flying robot being as perfect spy is certainly possible. However, there are other benevolent uses [2]. MAVs can be used for conducting aerial surveillance and reconnaissance [3], which are more difficult for ground vehicles. Their small size and agility in three dimensional space make them available for rescue in the aftermath of an earthquake and in other hazardous environments polluted by nuclear, biological, and chemical weapons [1].

According to their flight mechanisms, MAVs can be categorized into three classes: fixed-winged, rotary-winged, and flapping-winged. In order to stay in the air, fixed-winged MAVs need to keep translating in the sky, which is not desirable in confined space. Flapping-winged robots are still in their nascent stage, although one attempt has succeeded to make a life size insect robot just to generate enough thrust by its artificial stick-thin wings [2]. On the other hand, rotary-winged MAVs have the ability of Vertical Take-Off and Landing (VTOL) and motionless hover in the sky, which are desirable features in restricted indoor flight.

Among the rotary-winged MAVs, coaxial double rotor configuration has the further advantage of being compact over the traditional configuration which has a long tail. Of this type, Epson Company and Chiba University in Japan jointly developed $\mu$FR (Micro Flying Robot) [1] [4] [5], which has successfully demonstrated autonomous hovering flight; ETH Zurich’s CoaX series also showed hover ability [6].
Traditional VTOL vehicles use swash plate to change the collective and cyclic pitch angles of the rotor in order to control the orientation of the vehicles. However, this complex mechanism is hard to apply on MAVs and future smaller-scale aerial vehicles. For coaxial double rotor configuration, one alternative is to use the Center of Gravity (CoG) shifting mechanism which are adopted by both $\mu$FR and CoaX [4] [5] [6].

The development of analytic models for coaxial double rotor MAVs with CoG shifting mechanism is rare in the literature. For example, the $\mu$FR’s control systems are based on identified models due to model complexity [1]. The PID controller for CoaX is not model-based [6]. The contribution of this thesis is to give two analytic models from first principles for a double-rotor micro flying robot called C-fly which adopts CoG shifting mechanism. The first model treats C-fly as a quasi rigid body where Euler’s Law for the rotational motion of a rigid body is used but the variation of the moments of inertia due to the change of the CoG is captured. In essence, C-fly is not a rigid body but a multi rigid body system. The second model alleviates the heavy assumptions of the previous one and uses a variational principle of mechanics.

The conceptual design of C-fly is presented in Chapter 2. The modeling of C-fly as a quasi rigid body is described in detail in Chapter 3. A LQR control system is designed in Chapter 4 which stabilizes the attitude of the hovering C-fly. Chapter 5 presents the modeling of C-fly as a multi rigid-body system based on which a LQR control system is studied in Chapter 6. In the conclusion, future research is discussed.
2. CONCEPTUAL DESIGN OF C-FLY

The coaxial double rotor micro flying robot in later discussion is named C-fly. It consists of four main body parts: the upper rotor, the lower rotor, the fuselage and the CoG shifting mechanism. Fig. 2.1(a) and 2.1(b) show the top and lateral views of the upper rotor, respectively. And Fig. 2.1(c) and 2.1(d) show the top and lateral views of the lower rotor, respectively. When the two is assembled together, \(^1\text{C}\) stands for the Calumet campus of Purdue University which is near the city of Chicago. The author himself is from China. It is good to really see it fly up some day.
the top rotor’s solid shaft is inserted into the lower rotor’s hollow shaft to make the
two rotors coaxial.

The fuselage is divided into three rooms as shown in Fig. 2.2. The top room houses
the two actuators for driving the two rotors. The room in the middle holds the inertial
sensors, the batteries, and the micro controller. The CoG shifting mechanism resides
in the room on the bottom.

Instead of positioning the CoG by rotational actuator as that in [6], C-fly has a
two degree of freedom mass block which is confined to a plane parallel to the vehicle’s
chassis and moves along two perpendicular directions. Fig. 2.3 illustrates this linear
mechanism which mimics the blade’s motion of a lathe.

To put them all together, C-fly is illustrated in Fig. 2.4.
Fig. 2.4. Illustration of C-fly
3. MODELING OF C-FLY AS A QUASI RIGID BODY

The reference inertial frame, which is fixed on the earth, is assigned as \( OXYZ \) shown in Fig. 3.1. C-fly’s position is denoted by the Cartesian coordinates \((X, Y, Z)\) of its center of mass in \( OXYZ \). Let \( oxyz \) be the local reference frame fixed on C-fly with the origin \( o \) coinciding with the center of mass of C-fly. By aerospace industry conventions, the x-axis comes out of the anterior of the vehicle; the y-axis points to the right of the vehicle; the z-axis is along the axisymmetric axis pointing downward to the bottom of the vehicle. C-fly’s orientation is denoted by the Tait-Brian (roll, pitch, yaw) angles \((\phi, \theta, \psi)\). From the current orientation of \( oxyz \) in \( OXYZ \), the

Fig. 3.1. Reference \((OXYZ)\) and local \((oxyz)\) coordinate frames and the four components of C-fly: 1. upper rotor, 2. lower rotor, 3. fuselage, and 4. mass block.
vehicle rotates by $\phi$ about its x-axis, then by $\theta$ about the consequent y-axis, and by $\psi$ about the following z-axis, then $\textit{xyz}$ is parallel to $\textit{OXYZ}$.

The four basic parts – the upper rotor, the lower rotor, the fuselage, and the mass block – each is treated as a rigid body and they are assigned the numbers 1, 2, 3, and 4, respectively as shown in Fig. 3.1. It is assumed that 1 rotates counterclockwise and 2 rotates clockwise.

The angular velocity $\vec{\omega} = [\omega_x, \omega_y, \omega_z]^T$ is defined with respect to the standard basis $e_x, e_y, e_z$ of $\textit{xyz}$. The kinematics between $[\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$ and $[\omega_x, \omega_y, \omega_z]^T$ is shown below [7].

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \theta \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\tag{3.1}
\]

The values of $[\omega_x, \omega_y, \omega_z]^T$ can be acquired from the gyroscopes mounted on the C-fly. The values of $[\phi, \theta, \psi]^T$ are updated using (3.1). Any vector $[x, y, z]^T$ expressed with respect to the basis of $\textit{xyz}$ are transformed to the vector $[X, Y, Z]^T$ in the reference frame $\textit{OXYZ}$ as follows.

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = 
\text{Rot}(z, \psi)\text{Rot}(y, \theta)\text{Rot}(x, \phi)
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\tag{3.2}
\]

The model of C-fly is based on the following assumptions.

i) The components 1, 2, 3, and 4 are precisely engineered to be axisymmetric. Initially, 4 is located just below the shaft, making the products of inertia of C-fly all be zeros.

ii) The axiymmetry is destroyed if 4 shifts relative to the fuselage and the products of inertia will not be zeros. However, since the displacements are small and the products are much smaller than the moments of inertia, the products of inertia are approximately zero. Although this point needs further justification, it is assumed that products of inertia are always zero in order to simplify the analysis.
iii) Under any circumstance, each blade of the upper and lower rotors does not have flapping, leading-lagging, or feathering motions. The tip path plane is always perpendicular to the shaft.

iv) The thrust coefficients and the drag coefficients of the two rotors are constants.

v) Although the dynamical equations for orientation are derived based on a single rigid body, they are used here to model the amalgam of multiple rigid bodies.

3.1 Rotational Dynamics

The dynamical equation describing C-fly’s rotational motion is shown below

\[ \mathbf{M} = (\dot{\mathbf{H}})_{oxyz} + \bar{\omega} \times \mathbf{H}, \]  

(3.3)

where \( \mathbf{M} = [M_x, M_y, M_z]^T \) is the external moment exerted on C-fly; \( \mathbf{H} = [H_x, H_y, H_z]^T \) is the angular momentum of C-fly; \( (\dot{\mathbf{H}})_{oxyz} \) is the rate of change of the absolute angular momentum with respect to the origin of \( oxyz \); and \( \bar{\omega} \) is the absolute angular...
velocity of C-fly. The above vector decompositions are with respect to the standard basis $e_x, e_y, e_z$ of $oxyz$.

The three angular momenta are

\[
H_x = I_{xx} \omega_x, \\
H_y = I_{yy} \omega_y, \\
H_z = I_{zz} + J(\Omega_1 - \Omega_2),
\]

(3.4)

where $I_{xx}$ and $I_{yy}$ are the moments of inertia of C-fly around the local $x$ axis and $y$ axis, respectively. $I_{zz}$ is the moment of inertia of 3 and 4 about the local $z$ axis. $J$ is the moment of inertia of 1 or 2 about the local $z$ axis. $\Omega_1$ is the angular speed of 1 and $\Omega_2$ is the angular speed of 2. We define that when $\Omega_1 > 0$, the upper rotor rotates counter clockwise; when $\Omega_2 > 0$, the lower rotor rotates clockwise.

As 4 shifts in 3, $I_{xx}, I_{yy}, I_{zz}$, and $J$ vary depending on the position of the mass block (see Fig. 3.2.). Therefore, the time derivative of each angular momentum component is not the corresponding moment of inertia times the angular acceleration.

By the parallel axis theorem, we have

\[
I_{xx} = I_{xx}^{1,2,3} + I_{xx}^4 + \frac{Mm}{M+m}(h^2 + b^2), \\
I_{yy} = I_{yy}^{1,2,3} + I_{yy}^4 + \frac{Mm}{M+m}(h^2 + a^2), \\
I_{zz} = I_{zz}^3 + I_{zz}^4 + \frac{m_f m + M^2}{(M+m)^2}(a^2 + b^2), \\
J = J^S + \frac{m_f m^2(a^2 + b^2)}{(M+m)^2},
\]

(3.5)

where $M$ is the total mass of 1, 2, and 3; $m$ is the mass of 4; $m_f$ is the mass of 3; $m_r$ is the mass of 1 or 2; $I_{ii}^{j_1 j_2 \cdot \cdot \cdot}$ stands for the moment of inertia about the local $i$ axis ($i = x, y, z$) of the components $j_1, j_2, \cdot \cdot \cdot$ where $j_k \in \{1, 2, 3, 4\}$; $J^S$ is the moment of inertia of 1 or 2 about the shaft. In addition, $(a, b)$ is the coordinate of 4 relative to the center of mass of 1, 2 and 3 along the local $x$ and $y$ axes, respectively.
It is assumed that the drag moment on the fuselage is negligible, then the moments exerted on C-fly are ascribed to the thrusts of the two rotors and their drag forces as shown below.

\[
\begin{align*}
M_x &= (T_1 + T_2) \frac{m}{M+m} b, \\
M_y &= - (T_1 + T_2) \frac{m}{M+m} a, \\
M_z &= - \text{sgn}(\Omega_1) D_1 + \text{sgn}(\Omega_2) D_2,
\end{align*}
\]

(3.6)

where \( T_1 \) and \( T_2 \) are the thrust forces generated by the upper and lower rotors, respectively. \( D_1 \) is the drag moment on the upper rotor, and \( D_2 \) is the drag moment on the lower rotor. According to [9], through dimensional analysis, \( T_i = k T_i \Omega_i^2 \), where \( k \) is the thrust coefficient of \( i \); \( D_i = k D_i \Omega_i^2 \), where \( k \) is the drag coefficient. In the previous definitions, \( i = 1, 2 \); \( r \) is the identical radius of 1 or 2; and \( \rho \) is the density of the air.

### 3.2 Translational Dynamics

Translational dynamics is derived by applying Newton’s second law

\[
\mathbf{F} = (M + m) \ddot{\mathbf{r}},
\]

(3.7)

where \( \mathbf{F} = [F_x, F_y, F_z]^T \) is the external force and \( \mathbf{r} = [X, Y, Z]^T \) is the position vector of the center of mass of C-fly.

Rotor thrusts, drag forces and gravity contribute to the external forces as indicated below.

\[
\begin{align*}
F_x &= - (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)(T_1 + T_2) \\
&\quad - C_{D}^{\text{trans}} \rho S \dot{X} \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}, \\
F_y &= - (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi)(T_1 + T_2) \\
&\quad - C_{D}^{\text{trans}} \rho S \dot{Y} \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}, \\
F_z &= (M + m) g - \cos \phi \cos \theta(T_1 + T_2) \\
&\quad - C_{D}^{\text{trans}} \rho S \dot{Z} \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2},
\end{align*}
\]

(3.8)

where thrusts \( T_1 \) and \( T_2 \) are projected onto the standard basis of OXYZ by applying (3.2); \( C_{D}^{\text{trans}} \) is the translational drag coefficient and \( S \) is the equivalent cross section area of C-fly.
3.3 Rotors and Mass Block Dynamics

Through model identification, [10] has shown that it is sufficient to model the rotor driven by a brushless DC motor by a first order system. Let $\Omega_R$ be the reference rotor angular speed, and $\Omega$ be the actual rotor angular speed. Assume 1 and 2 have the same dynamics, namely

$$\frac{\Omega(s)}{\Omega_R(s)} = \frac{k_r}{\tau_r s + 1},$$

(3.9)

where $k_r$ is the steady state gain, and $\tau_r$ is the time constant.

The dynamics of the mass block for the two perpendicular directions are assumed to be two second order critically damped systems. Let $d_R$ be the reference position of the mass block along one direction and $d$ be the actual position. Then, we have

$$\frac{d(s)}{d_R(s)} = \frac{k_b}{(\tau_b s + 1)^2},$$

(3.10)

where $k_b$ is the steady state gain and $\tau_b$ is the time constant.

3.4 State Space Model

To design the control system and simulate the dynamics, the state space model is obtained. The system has four control inputs and eighteen state variables. The four control inputs are

$$u_1 = \Omega_1 R, \quad u_2 = \Omega_2 R, \quad u_3 = a, \quad u_4 = b,$$

(3.11)

where $\Omega_1 R$ and $\Omega_2 R$ are the reference speed of the upper and lower rotor, respectively, and $(a, b)$ is the reference position of the mass block.

The first six state variables defined in (3.12) describe the 1, 2 and 4’s dynamics.

$$x_1 = \Omega_1, \quad x_2 = \Omega_2, \quad x_3 = a, \quad x_4 = \dot{a}, \quad x_5 = b, \quad x_6 = \dot{b}$$

(3.12)

The next six state variables are assigned to the rotational dynamics. It is observed in (3.1) that the matrix relating $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ and $(\omega_x, \omega_y, \omega_z)$ depends on $\phi$ and $\theta$ only.
Since φ and θ are small when C-fly is in either vertical or horizontal flight, this matrix can be approximated by the identity matrix. Thus, given the same initial conditions, \( \int_0^t \omega_x(\zeta) d\zeta, \int_0^t \omega_y(\zeta) d\zeta, \int_0^t \omega_z(\zeta) d\zeta \) approximates φ, θ, ψ, respectively and they are selected as \( x_7, x_9 \) and \( x_{11} \). In addition, the angular momenta are selected as state variables \( x_8, x_{10}, \) and \( x_{12} \) as shown in (3.13).

\[
\begin{align*}
    x_7 &= \phi \\
    x_8 &= H_x = I_{xx} \dot{\phi} \\
    x_9 &= \theta \\
    x_{10} &= H_y = I_{yy} \dot{\theta} \\
    x_{11} &= \psi \\
    x_{12} &= H_z = I_{zz} \dot{\psi} + J(\Omega_1 - \Omega_2)
\end{align*}
\]

The six state variables for translational dynamics are defined as

\[
\begin{align*}
    x_{13} &= X \\
    x_{14} &= \dot{X} \\
    x_{15} &= Y \\
    x_{16} &= \dot{Y} \\
    x_{17} &= Z \\
    x_{18} &= \dot{Z}
\end{align*}
\]

The state space model is shown in (3.16), where \( k_{Trans}^{Trans} = C_{Trans}^{Trans} \rho S \) and

\[
\begin{align*}
    I_{xx} &= I_{xx}^{1,2,3} + I_{xx}^4 + \frac{Mm}{M+m}(h^2 + x_5^2) \\
    I_{yy} &= I_{yy}^{1,2,3} + I_{yy}^4 + \frac{Mm}{M+m}(h^2 + x_3^2) \\
    I_{zz} &= I_{zz}^3 + I_{zz}^4 + \frac{m(m_I+m+M_2)}{(M+m)^2}(x_3^2 + x_5^2) \\
    J &= JS + \frac{m_v m^2 (x_3^2 + x_5^2)}{(M+m)^2}
\end{align*}
\]
\[\begin{align*}
\dot{x}_1 &= \frac{k_r}{\tau_r}u_1 - \frac{1}{\tau_r}x_1 \\
\dot{x}_2 &= \frac{k_r}{\tau_r}u_2 - \frac{1}{\tau_r}x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{k_{\tau}}{\tau_r}u_3 - \frac{2}{\tau_b}x_4 - \frac{1}{\tau_b}x_3 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \frac{k_{\tau}}{\tau_r}u_4 - \frac{2}{\tau_b}x_6 - \frac{1}{\tau_b}x_5 \\
\dot{x}_7 &= x_8/I_{xx} \\
\dot{x}_8 &= x_5(k_{T_1}x_1^2 + k_{T_2}x_2^2)m/(m + M) \\
&\quad + (I_{yy} - I_{zz})(x_{10}/I_{yy})(x_{12} - J(x_1 - x_2))/I_{zz} \\
&\quad - J(x_1 - x_2)x_{10}/I_{yy} \\
\dot{x}_9 &= x_{10}/I_{yy} \\
\dot{x}_{10} &= -x_3(k_{T_1}x_1^2 + k_{T_2}x_2^2)m/(m + M) \\
&\quad + (I_{zz} - I_{xx})(x_8/I_{xx})(x_{12} - J(x_1 - x_2))/I_{zz} \\
&\quad + J(x_1 - x_2)x_8/I_{xx} \\
\dot{x}_{11} &= (x_{12} - J(x_1 - x_2))/I_{zz} \\
\dot{x}_{12} &= -\text{sgn}(x_1)k_{D_1}x_1^2 + \text{sgn}(x_2)k_{D_2}x_2^2 \\
\dot{x}_{13} &= x_{14} \\
\dot{x}_{14} &= -(\cos x_7 \sin x_9 \cos x_{11} + \sin x_7 \sin x_{11}) \\
&\quad (k_{T_1}x_1^2 + k_{T_2}x_2^2)/(M + m) \\
&\quad -k_{D_1}^{\text{trans}}x_{14}\sqrt{x_{14}^2 + x_{16}^2 + x_{18}^2}/(M + m) \\
\dot{x}_{15} &= x_{16} \\
\dot{x}_{16} &= -(\cos x_7 \sin x_9 \sin x_{11} - \sin x_7 \cos x_{11}) \\
&\quad (k_{T_1}x_1^2 + k_{T_2}x_2^2)/(M + m) \\
&\quad -k_{D_1}^{\text{trans}}x_{16}\sqrt{x_{14}^2 + x_{16}^2 + x_{18}^2}/(M + m) \\
\dot{x}_{17} &= x_{18} \\
\dot{x}_{18} &= g - \cos x_7 \cos x_9(k_{T_1}x_1^2 + k_{T_2}x_2^2)/(M + m) \\
&\quad -k_{D_1}^{\text{trans}}x_{18}\sqrt{x_{14}^2 + x_{16}^2 + x_{18}^2}/(M + m)
\end{align*}\]
4. ATTITUDE STABILIZATION BASED ON THE FIRST MODEL

4.1 Control System Design

C-fly is by nature an unstable system. Small disturbance of the mass block’s position generates thrust moment that causes the C-fly to rotate significantly if it is not controlled. Actually, the system model is not valid any more if the system digress too much from the hovering mode and it is necessary to stabilize it about the hovering equilibrium point. In this paper, the Linear Quadratic Regulator (LQR) is used as a controller to stabilize the attitude based on the developed linearized model.

Making the right hand sides of (3.16) equal to zero and considering the physical meaning of “hover” yield the equilibrium point as follows.

\[
\begin{align*}
   u_1^* &= x_1^*/k_r, \quad u_2^* = x_2^*/k_r, \quad u_3^* = 0, \quad u_4^* = 0, \\
   x_1^* &= \frac{g(M+m)}{k_{T_1} + k_{T_2} k_{D_1}/k_{D_2}}, \quad x_2^* = \sqrt{\frac{g(M+m)}{k_{T_1} k_{D_2} / k_{D_1} + k_{T_2}}}, \\
   x_3^* &= 0, \quad x_4^* = 0, \quad x_5^* = 0, \quad x_6^* = 0, \\
   x_7^* &= 0, \quad x_8^* = 0, \quad x_9^* = 0, \\
   x_{10}^* &= 0, \quad x_{11}^* = 0, \quad x_{12}^* = J_S (x_1^* - x_2^*), \\
   x_{13}^* &= 0, \quad x_{14}^* = 0, \quad x_{15}^* = 0, \\
   x_{16}^* &= 0, \quad x_{17}^* = 0, \quad x_{18}^* = 0.
\end{align*}
\] (4.1)

To stabilize the attitude, let \( \Delta x = [x_1 - x_1^*, x_2 - x_2^*, \ldots, x_{12} - x_{12}^*]^T \) and \( \Delta u = [u_1 - u_1^*, \ldots, u_4 - u_4^*]^T \). Then the linearized model is

\[
\Delta x = A\Delta x + B\Delta u,
\] (4.2)

where \( A \) and \( B \) are given as follows.
\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

where

\[ A_{11} = \begin{bmatrix} -\frac{1}{\tau_r} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\tau_r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau_b} & -\frac{2}{\tau_b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{\tau_b} & -\frac{2}{\tau_b} \end{bmatrix} \]

\[ A_{12} = O_{6 \times 6} \]

\[ A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & mg & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -mg & 0 & 0 & 0 \\ \frac{J^S}{I_{xx} + I_{xz}} & \frac{J^S}{I_{xx} + I_{xz}} & 0 & 0 & 0 & 0 \\ -2kD_1x_1^* & 2kD_2x_2^* & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ A_{22} = \begin{bmatrix} 0 & \frac{1}{I_{xx}|b=0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{J^S(x_1^* - x_2^*)}{I_{yy}|a=0} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I_{yy}|a=0} & 0 & 0 \\ 0 & \frac{J^S(x_1^* - x_2^*)}{I_{xx}|b=0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I_{xx} + I_{xz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{I_{xx} + I_{xz}} & 0 \end{bmatrix} \]
\[
B_1 = \begin{bmatrix}
\frac{k_r}{\tau_r} & 0 & 0 & 0 \\
0 & \frac{k_r}{\tau_r} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{k_b}{\tau_b} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{k_b}{\tau_b}
\end{bmatrix} \quad B_2 = O_{6 \times 4}
\]

Although only eight of the twelve states of the above system are controllable, the system is stabilizable. The LQR controller applied to the linearized model minimizes the performance index

\[
J(u) = \int_0^\infty (\Delta x^T Q \Delta x + \Delta u^T R \Delta u) dt, \tag{4.3}
\]

where \( Q \) and \( R \) are the weighting matrices. The control input using full state feedback is

\[
\Delta u = -K \Delta x. \tag{4.4}
\]

In (4.4), \( K = R^{-1}B^T S \), where \( S \) is the solution of the algebraic Riccati equation

\[
A^T S + S A - S B R^{-1} B^T S + Q = 0. \tag{4.5}
\]

4.2 Simulation Results

The control system (3.16) with (4.4) is simulated in MATLAB and Simulink. Typical values of the parameters as shown in Table 4.1 are used. The moments of inertia are computed based on the assumption at the beginning of Chapter 3.

The initial value for roll, pitch, and yaw angles are all set to 0.1 rad (5.73°) and all the other values for the state-variables are set to zeros. The measurements of the three angles are corrupted by zero mean band-limited Guassian noises with correlation time 0.1 second and covariance 0.1 rad. The duration of the simulation is 10 seconds.

If the weighting matrices \( Q \) and \( R \) in (4.3) are identity matrices, then the results are shown in Fig. 4.1. and Fig. 4.2. It can be observed that the yaw angle \( \psi \)
Table 4.1
Simulation Parameters for C-fly

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_r$</td>
<td>0.1 s</td>
<td>$C_T^{Trans}$</td>
<td>1.2</td>
</tr>
<tr>
<td>$\tau_b$</td>
<td>0.05 s</td>
<td>$h$</td>
<td>0.03 m</td>
</tr>
<tr>
<td>$k_r$</td>
<td>0.9</td>
<td>$m_r$</td>
<td>0.05 kg</td>
</tr>
<tr>
<td>$k_b$</td>
<td>0.9</td>
<td>$m_f$</td>
<td>0.2 kg</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1.204 kg/m³</td>
<td>$m$</td>
<td>0.1 kg</td>
</tr>
<tr>
<td>$r$</td>
<td>0.1 m</td>
<td>$M$</td>
<td>0.3 kg</td>
</tr>
<tr>
<td>$S$</td>
<td>0.0157 m²</td>
<td>$J^S$</td>
<td>$2.5 \times 10^{-4}$ kg · m²</td>
</tr>
<tr>
<td>$C_{T_1}$</td>
<td>0.1</td>
<td>$I_{xx}^{1,2,3} = I_{yy}^{1,2,3}$</td>
<td>$8.73 \times 10^{-4}$ kg · m²</td>
</tr>
<tr>
<td>$C_{T_2}$</td>
<td>0.07</td>
<td>$I_{xx}^{4} = I_{yy}^{4}$</td>
<td>$6.67 \times 10^{-6}$ kg · m²</td>
</tr>
<tr>
<td>$C_{D_1}$</td>
<td>0.02</td>
<td>$I_{zz}^{3}$</td>
<td>$2.5 \times 10^{-4}$ kg · m²</td>
</tr>
<tr>
<td>$C_{D_2}$</td>
<td>0.016</td>
<td>$I_{zz}^{4}$</td>
<td>$6.66 \times 10^{-6}$ kg · m²</td>
</tr>
</tbody>
</table>

undergoes large overshoot with peak value 220° compared with the bound ±30° for $\phi$ and $\theta$. After its transient process the CoG shifting mechanism plays the major role in stabilizing the attitude of C-fly.

To moderate the transient response of $\psi$, the identity weighting function $Q$’s element $Q(12,12)$ is changed to 10. In this case, the peak value of the overshoot is reduced to 90° compared with 220° in the previous case; the transient time for $\psi$ is reduced from 4 seconds to 1 second as shown in Fig. 4.4. Notice that instead of being almost constants as in the previous case, the upper and lower rotor speeds varies slightly according to the noise effect as shown in Fig. 4.3.
Fig. 4.1. Rotor speeds and mass block position
Fig. 4.2. Rotational motion
Fig. 4.3. Rotor speeds and mass block position
Fig. 4.4. Rotational motion
5. MODELING OF C-FLY AS A MULTI RIGID-BODY SYSTEM

In this chapter, Section 5.2 and 5.3 are based on the theoretical work of Sangbum Cho [11].

5.1 Hamilton’s Principle

According to D’Alembert’s principle, the infinitesimal quantity - the total virtual work done by the impressed and inertial forces - is equal to zero. By integrating this quantity with respect to time and assuming that the variation of the variables all vanish at the end points, we get the generalized Hamilton’s principle:

\[ \int_{t_0}^{t_f} (\delta W + \delta T) dt = 0, \]  

(5.1)

where \( \delta W \) is the infinitesimal virtual work done by the impressed forces and \( \delta T \) is the resultant variation of the kinetic energy.

If we assume that all the impressed forces are derivable from a single potential function \( V \), or \( \delta W = -\delta V \), then we get

\[ \int_{t_0}^{t_f} \delta L dt = 0, \]  

(5.2)

where \( L = T - V \) is called the Lagrangian. Notice that we cannot arbitrarily pull the \( \delta \) to be in front of the integral sign. But if the system is holonomic, we can do this and we get

\[ \delta \int_{t_0}^{t_f} L dt = 0. \]  

(5.3)

This is the Hamilton’s Principle in the narrow sense. Nonetheless, this principle is most useful since it is in the form of a variational problem. We can apply calculus of variation to establish the equations of motion for the system.
Thus, the Lagrangian, or the difference between the kinetic energy and the potential energy is needed to model a mechanical system.

### 5.2 The Lagrangian

In the modeling of C-fly based on variational principles, the four body parts are no longer treated equally as in the single quasi rigidbody case. Instead, we designate the fuselage to be the base body, to which the other three parts are attached. Thereafter, we are interested in the position vector $x$ of the center of mass of the base body in the inertial frame instead of that of the center of mass of the whole system. In addition, the angular velocity $\omega$ of the base body in the body fixed frame is of interest to us.

Let the indexing set be $I = \{u, l, b\}$, where $u$, $l$ and $b$ represent the upper rotor, the lower rotor and the mass block respectively. Let $x_i$ be the inertial position vector of $i$, $\omega_i$ be the angular velocity of $i$ in the body fixed frame, $\rho_i$ be the relative position of the center of mass of body $i$ with respect to the center of mass of the base body in the body fixed frame, $J_i$ be the inertia matrix of $i$ and $m_i$ be the mass of $i$, where $i \in I$. The generalized coordinates are $q = \left[ \int_0^t \Omega_u(\xi)d\xi, \int_0^t \Omega_l(\xi)d\xi, a, b \right]^T$, where $\Omega_u$ and $\Omega_l$ are the upper and lower rotor speeds respectively, and $[a, b]^T$ is the position vector of the mass block relative to the projection of the center of mass of the base body on the plane that the mass block shifts. Notice that the $\rho_i$’s are functions of $q$.

Let $J$ and $m$ be the inertia matrix and the mass of the base body, respectively. The kinetic energy of the four-body system is

$$ T = \frac{1}{2} m \dot{x}^T \dot{x} + \frac{1}{2} \omega^T J \omega + \sum_{i \in I} \left( \frac{1}{2} m_i \dot{x}_i^T \dot{x}_i + \frac{1}{2} \omega_i^T J_i \omega_i \right). \quad (5.4) $$
Let $R$ be the rotation matrix from the body fixed frame to the inertial frame, then the expression for $\dot{x}_i$ is given by

$$
\dot{x}_i = \frac{d}{dt}(x + R\rho_i)
= \dot{x} + R\left(\omega \times \rho_i + \left[\frac{\partial \rho_i}{\partial q}\right] \dot{q}\right)
= \dot{x} + R\left(\dot{\omega} \rho_i + \left[\frac{\partial \rho_i}{\partial q}\right] \dot{q}\right)
= \dot{x} + R\left(-\dot{\rho}_i \omega + \left[\frac{\partial \rho_i}{\partial q}\right] \dot{q}\right),
$$

(5.5)

where the hat map $\hat{\cdot}: \mathbb{R}^3 \to so(3)$ is defined by

$$
\begin{bmatrix}
    0 & -x_3 & x_2 \\
    x_3 & 0 & -x_1 \\
    -x_2 & x_1 & 0
\end{bmatrix}.
$$

(5.6)

The expression for $\omega_i$ is given by

$$
\omega_i = \omega + C_i(q)\dot{q},
$$

(5.7)

where $C_i(q)$ is a defined constraint function for the angular motion of body $i$. 
Substitute (5.5) and (5.7) into (5.4) and notice that $R^T R = I$, we get

\[
T = \frac{1}{2} m \ddot{x}^T \dddot{x} + \frac{1}{2} \omega^T J \omega \\
+ \sum_{i \in I} \left[ \frac{1}{2} m_i \left( \dddot{x} - R \dddot{\hat{\rho}} \omega + R \left[ \frac{\partial \rho_i}{\partial q} \right] \dot{q} \right)^T \left( \dddot{x} - R \dddot{\hat{\rho}} \omega + R \left[ \frac{\partial \rho_i}{\partial q} \right] \dot{q} \right) \right] \\
+ \frac{1}{2} \left( \omega + C_i(q) \dot{q} \right)^T J_i \left( \omega + C_i(q) \dot{q} \right) \\
= \frac{1}{2} \left( m + \sum_{i \in I} m_i \right) \dddot{x}^T \dddot{x} + \frac{1}{2} \omega^T \left[ J + \sum_{i \in I} (m_i \dddot{\hat{\rho}}_i + J_i) \right] \omega \\
+ \frac{1}{2} q^T \sum_{i \in I} \left( m_i \left[ \frac{\partial \rho_i}{\partial q} \right]^T \left[ \frac{\partial \rho_i}{\partial q} \right] + C_i^T(q) J_i C_i(q) \right) \dot{q} \\
- \frac{1}{2} \dddot{x}^T \sum_{i \in I} m_i R \dddot{\hat{\rho}}_i \omega - \frac{1}{2} \left( \dddot{x}^T \sum_{i \in I} m_i R \dddot{\hat{\rho}}_i \right) \omega \\
+ \frac{1}{2} \dddot{x}^T \sum_{i \in I} m_i R \left[ \frac{\partial \rho_i}{\partial q} \right] \dot{q} + \frac{1}{2} \left( \dddot{x}^T \sum_{i \in I} m_i R \left[ \frac{\partial \rho_i}{\partial q} \right] \right)^T \dot{q} \\
+ \frac{1}{2} \omega^T \sum_{i \in I} \left( -m_i \dddot{\hat{\rho}}_i \left[ \frac{\partial \rho_i}{\partial q} \right] + J_i C_i(q) \right) \dot{q} \\
+ \frac{1}{2} \left( \omega^T \sum_{i \in I} \left( -m_i \dddot{\hat{\rho}}_i \left[ \frac{\partial \rho_i}{\partial q} \right] + J_i C_i(q) \right) \right)^T.
\]

Let

\[
M = \left( m + \sum_{i \in I} m_i \right) I_{3 \times 3} \\
J = J + \sum_{i \in I} (m_i \dddot{\hat{\rho}}_i + J_i) \\
M_i = \sum_{i \in I} \left( m_i \left[ \frac{\partial \rho_i}{\partial q} \right]^T \left[ \frac{\partial \rho_i}{\partial q} \right] + C_i^T(q) J_i C_i(q) \right) \\
K = - \sum_{i \in I} m_i \dddot{\hat{\rho}}_i \\
B_t = \sum_{i \in I} m_i \left[ \frac{\partial \rho_i}{\partial q} \right] \\
B_r = \sum_{i \in I} \left( -m_i \dddot{\hat{\rho}}_i \left[ \frac{\partial \rho_i}{\partial q} \right] + J_i C_i(q) \right),
\]

(5.9)
then the kinetic energy can be written in a compact form

\[
T = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \omega \\ \dot{q} \end{bmatrix}^T \begin{bmatrix} \mathcal{M} & RK & RB_t \\ K^T R^T & J & B_r \\ B_i^T R^T & B_r^T & M_l \end{bmatrix} \begin{bmatrix} \dot{x} \\ \omega \\ \dot{q} \end{bmatrix}.
\]  
(5.10)

Taking the plane of \(OXY\) to be of zero gravitational potential energy, the gravitational potential energy of C-fly is

\[
V = -mg e_3^T x - \sum_{i \in I} m_i g e_3^T (R \rho_i + x).
\]  
(5.11)

Therefore, the Lagrangian for C-fly is

\[
L = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \omega \\ \dot{q} \end{bmatrix}^T \begin{bmatrix} \mathcal{M} & RK & RB_t \\ K^T R^T & J & B_r \\ B_i^T R^T & B_r^T & M_l \end{bmatrix} \begin{bmatrix} \dot{x} \\ \omega \\ \dot{q} \end{bmatrix} + mg e_3^T x + \sum_{i \in I} m_i g e_3^T (R \rho_i + x).
\]  
(5.12)

Let the distance from the upper rotor to \(o\) be \(l_1\), the distance from the lower rotor to \(o\) be \(l_2\), and the distance from \(o\) to the plane where the mass block moves on be \(h\).

Let the position of the mass block along \(x\)-axis and \(y\)-axis be \(a\) and \(b\), respectively. According to the definition of \(\rho\), we know that for C-fly,

\[
\rho_u = [0, 0, -l_1]^T,
\]  
(5.13)

\[
\rho_i = [0, 0, -l_2]^T,
\]  
(5.14)

and

\[
\rho_b = [a, b, h]^T.
\]  
(5.15)

To make the analysis simple, we assume the products of inertia defined at the CoG for each of the four rigid bodies are zero. Therefore, \(J\) and \(J_i\) are diagonal matrices

\[
J = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}, \quad J_i = \begin{bmatrix} I_{xx}^i & 0 & 0 \\ 0 & I_{yy}^i & 0 \\ 0 & 0 & I_{zz}^i \end{bmatrix}.
\]  
(5.16)
Then, we get
\[
\mathcal{J} = \begin{bmatrix}
I_1 & -abm_b & -ahm_b \\
-abm_b & I_2 & -bhm_b \\
-ahm_b & -bhm_b & I_3
\end{bmatrix},
\] (5.17)

where
\[
I_1 = I_{xx} + \sum_{i \in I} I^i_{xx} + l_1^2 m_u + l_2^2 m_l + (h^2 + b^2) m_b,
\] (5.18)

\[
I_2 = I_{yy} + \sum_{i \in I} I^i_{yy} + l_1^2 m_u + l_2^2 m_l + (h^2 + a^2) m_b,
\] (5.19)

and
\[
I_3 = I_{zz} + \sum_{i \in I} I^i_{zz} + (a^2 + b^2) m_b.
\] (5.20)

For C-fly, also notice that
\[
C_u(q) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix},
C_l(q) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix},
\] (5.21)

and
\[
C_b(q) = O_{3 \times 4}.
\] (5.22)

Thus,
\[
\mathcal{M} = \begin{bmatrix}
I^u_{zz} & 0 & 0 & 0 \\
0 & I^l_{zz} & 0 & 0 \\
0 & 0 & m_b & 0 \\
0 & 0 & 0 & m_b
\end{bmatrix},
\] (5.23)

\[
\mathcal{K} = \begin{bmatrix}
0 & -l_1 m_u - l_2 m_l + h m_b & -b m_b \\
l_1 m_u + l_2 m_l - h m_b & 0 & a m_b \\
0 & a m_b & 0
\end{bmatrix},
\] (5.24)

\[
\mathcal{B}_l = \begin{bmatrix}
0 & 0 & m_b & 0 \\
0 & 0 & 0 & m_b \\
0 & 0 & 0 & 0
\end{bmatrix},
\] (5.25)
and
\[ B_r = \begin{bmatrix} 0 & 0 & 0 & h m_b \\ 0 & 0 & -h m_b & 0 \\ F_u^{zz} & -F_i^{zz} & b m_b & -a m_b \end{bmatrix}. \] (5.26)

5.3 Equations of Motion

After deriving the Lagrangian, the equations of motion for the system are derived. According to Hamilton’s Principle, we have
\[ \delta \int_{t_0}^{t_f} L(x, \dot{x}, q, \dot{q}, \omega, R) dt = 0. \] (5.27)

Let us partition \( R \) as
\[ R^T = [R_1, R_2, R_3], \] (5.28)
then we have
\[ \int_{t_0}^{t_f} \left\{ \left( \frac{\partial L}{\partial x} \right)^T \delta x + \left( \frac{\partial L}{\partial \dot{x}} \right)^T \delta \dot{x} + \left( \frac{\partial L}{\partial q} \right)^T \delta q + \left( \frac{\partial L}{\partial \dot{q}} \right)^T \delta \dot{q} + \left( \frac{\partial L}{\partial \omega} \right)^T \delta \omega + \sum_{i=1}^{3} \left( \frac{\partial L}{\partial R_i} \right)^T \delta R_i \right\} dt = 0. \] (5.29)

The difficulties in deriving the equations of motion is to get the expressions for the variation of \( R \) and \( \omega \).

Recall that the hat map \( \hat{\cdot} : \mathbb{R}^3 \to so(3) \) is defined by
\[ \hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \]
where \( x = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \). The reason to introduce this map is to switch back and forth between matrix multiplication and cross product since
\[ x \times y = \hat{x}y. \] (5.30)
It can also be verified by the definition of $\hat{\cdot}$ and (5.30) that the following two properties hold for the hat map and we will use them later.

$$\hat{x}y = -\hat{y}x$$  \hspace{1cm} (5.31)

$$\hat{x} \times y = \hat{x}\hat{y} - \hat{y}\hat{x}$$  \hspace{1cm} (5.32)

The rotational kinematics is expressed in terms of the hat map of $\omega$ as follows

$$\dot{R} = R\hat{\omega}.$$  \hspace{1cm} (5.33)

Thus,

$$\hat{\omega} = R^{-1}\dot{R} = RT\dot{R},$$  \hspace{1cm} (5.34)

where $R^{-1} = R^T$ has been used because $R$ preserves the inner product, or

$$RR^T = I.$$  \hspace{1cm} (5.35)

Taking the time derivative of the above equation, we also get

$$\dot{R}R^T + R\dot{R}^T = 0,$$  \hspace{1cm} (5.36)

or

$$\dot{R}^T = -R^T\dot{R}R^T.$$  \hspace{1cm} (5.37)

This will be used later.

To get the variations of $\omega$ and $R$, we first introduce a new map $\zeta : \mathbb{R} \rightarrow \mathbb{R}^3$ defined by

$$\dot{\zeta} = R^T\delta R.$$  \hspace{1cm} (5.38)

Thus, $\zeta$ is related to the variation of $R$ by the hat map. In addition, we require that $\zeta$ vanishes at the end points. Hence, $\zeta$ is the counterpart of $\delta x$ and $\delta q$.

Taking the time derivative of (5.38) yields

$$\dot{\zeta} = \dot{R}^T\delta R + R^T\delta \dot{R}$$
$$= -R^T\dot{R}R^T\delta R + R^T\delta \dot{R}$$
$$= -\hat{\omega}\zeta + R^T\delta \dot{R}.$$  \hspace{1cm} (5.39)
Thus, the variation of $\dot{\omega}$ is

$$
\delta \omega = \delta(R^T \dot{R})
= \delta R^T \dot{R} + R^T \delta \dot{R}
= -R^T \delta RR^T \dot{R} + R^T \delta \dot{R}
= -\dot{\hat{\zeta}} + \omega \hat{\zeta} - \hat{\zeta} \omega
$$

(5.40)

$$
= \dot{\hat{\zeta}} + \omega \times \zeta,
$$

where we have used the fact that the order of the operations of variation, hat mapping and time derivative can be interchanged. Hence, we have

$$
\delta \omega = \dot{\zeta} + \omega \times \zeta.
$$

(5.41)

If we let $e_1, e_2, e_3$ be the standard basis, then we have

$$
R_i = R^T e_i.
$$

(5.42)

Therefore, the variations of $R_i, i = 1, 2, 3$ are obtained as follows

$$
\delta R_i = \delta(R^T e_i) = -R^T \delta RR^T e_i = -\dot{\hat{\zeta}} R_i = R_i \times \zeta.
$$

(5.43)

Consequently, (5.29) becomes

$$
\int_{t_0}^{t_f} \left\{ \left( \frac{\partial L}{\partial x} \right)^T \delta x + \left( \frac{\partial L}{\partial \dot{x}} \right)^T \delta \dot{x} + \left( \frac{\partial L}{\partial q} \right)^T \delta q + \left( \frac{\partial L}{\partial \dot{q}} \right)^T \delta \dot{q} + \left( \frac{\partial L}{\partial \omega} \right)^T (\dot{\zeta} + \omega \times \zeta) + \sum_{i=1}^{3} \left( \frac{\partial L}{\partial R_i} \right)^T (R_i \times \zeta) \right\} dt = 0.
$$

(5.44)

Remember that all the variations vanish at end points, thus we have

$$
\int_{t_0}^{t_f} \left( \frac{\partial L}{\partial \dot{x}} \right)^T \delta \dot{x} dt = \int_{t_0}^{t_f} \left( \frac{\partial L}{\partial \dot{x}} \right)^T d(\delta x)
= \left( \frac{\partial L}{\partial \dot{x}} \right)^T \delta x|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{dt}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right)^T \delta x dt
= - \int_{t_0}^{t_f} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \delta x dt.
$$

(5.45)
Similarly, we have
\[
\int_{t_0}^{t_f} \left( \frac{\partial L}{\partial \dot{q}} \right)^T \delta \dot{q} dt = - \int_{t_0}^{t_f} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T \delta q dt, \tag{5.46}
\]
\[
\int_{t_0}^{t_f} \left( \frac{\partial L}{\partial \omega} \right)^T \delta \dot{\zeta} dt = - \int_{t_0}^{t_f} \frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right)^T \delta \zeta dt, \tag{5.47}
\]
and
\[
\int_{t_0}^{t_f} \left( \frac{\partial L}{\partial R_i} \right)^T \delta \dot{\zeta} dt = - \int_{t_0}^{t_f} \frac{d}{dt} \left( \frac{\partial L}{\partial R_i} \right)^T \delta \zeta dt.
\]
Substituting the above four equations into (5.44) and noticing that
\[
A^T (B \times C) = A^T \hat{B} C = (\hat{B}^T \hat{A})^T C = (-\hat{B} A)^T C = (-B \times A)^T C = (A \times B)^T C,
\]
where \(A, B,\) and \(C\) are \(3 \times 3\) matrices, (5.44) becomes
\[
\int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right]^T \delta x dt + \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right]^T \delta q dt + \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial \omega} \times \omega + \sum_{i=1}^{3} \frac{\partial L}{\partial R_i} \times R_i - \frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) \right]^T \delta \zeta dt = 0. \tag{5.48}
\]
Since the variations \(\delta x, \delta q,\) and \(\zeta\) are arbitrary, we have the following equations of motion
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x} \right) - \frac{\partial L}{\partial x} = 0, \tag{5.49}
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) - \frac{\partial L}{\partial q} = 0, \tag{5.50}
\]
and
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) - \frac{\partial L}{\partial \omega} \times \omega - \sum_{i=1}^{3} \frac{\partial L}{\partial R_i} \times R_i = 0, \tag{5.51}
\]
where the first two equations are in the same form of the Euler-Lagrange equations. In general, all the three equations are named Euler-Lagrange-Poincaré equations.
Recall from Section 5.1 that the Hamilton’s principle we have used is based on the assumption that all the external forces are derivable from a single potential function. However, this is not true in most interesting cases. A modification of Hamilton’s principle gives the following equations of motion

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = \tau_T, \tag{5.52}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \tau_S, \tag{5.53}
\]

and

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) - \frac{\partial L}{\partial \omega} \times \omega - \sum_{i=1}^{3} \frac{\partial L}{\partial R_i} \times R_i = \tau_R, \tag{5.54}
\]

where \(\tau_T\) is the nongravitational external force acting on the multibody system expressed in the global inertial frame; \(\tau_S\) is the generalized forces (moments) acting on the auxiliary bodies expressed in the base body frame; \(\tau_R\) is the nongravitational external moment acting on the whole system (see Appendix A).

Equations (5.52), (5.53) and (5.54) are in their most general form. Next, we express each set of equations in detail.

By the previous derivations,

\[
\frac{\partial L}{\partial \dot{x}} = M \ddot{x} + RK \omega + RB \dot{q}, \tag{5.55}
\]

and

\[
\frac{\partial L}{\partial x} = - \frac{\partial V}{\partial x}. \tag{5.56}
\]

Recall that \(\dot{R} = R\dot{\omega}\), then by substituting the above two equations into (5.52),

\[
M \ddot{x} + R\dot{\omega}K \omega + R\dot{K} \omega + RK \omega + R\dot{\omega}B_i \dot{q} + R\dot{B}_i \dot{q} + RB_i \dot{q} + \frac{\partial V}{\partial x} = \tau_T. \tag{5.57}
\]

\(\tau_T\) is mainly contributed by the thrusts generated by the two rotors, which are proportional to the square of the angular velocity of the rotor and perpendicular to the rotor disk. According to [9], the thrust generated by a rotor is

\[T = C_T \rho \pi r^4 \Omega^2,\]
where $C_T$ is the dimensionless thrust coefficient; $\rho$ is the density of the air; $r$ is the identical length of each blade. If we define $K_T = C_T \rho \pi r^4$, then $T = K_T \Omega^2$. In addition, let us assume when exposed to the translational drag C-fly can be assumed to be a ball. Therefore, let us suppose that the drag force’s magnitude is $C_{D}^{\text{Trans}} \rho S |\dot{x}|^2$, where $C_{D}^{\text{Trans}}$ is the translational drag coefficient. Consequently,

$$
\tau_T = -R(K_{T_u} \Omega_u^2 + K_{T_l} \Omega_l^2)e_3 - C_{D}^{\text{Trans}} \rho S |\dot{x}|.
$$

Recall the expression for the gravitational potential energy (5.11), we get

$$
\frac{\partial V}{\partial x} = -(m + \sum_{i \in I} m_i) g e_3.
$$

Therefore, noticing that $\dot{B}_t = 0$ for C-fly, the translational equation of motion is

$$
M \ddot{x} + R \dot{\omega} K \omega + R K \dot{\omega} + R \dot{\omega} B_t \dot{q} + R B_t \ddot{q} = -R(K_{T_u} \Omega_u^2 + K_{T_l} \Omega_l^2) + (m + \sum_{i \in I} m_i) g e_3 - C_{D}^{\text{Trans}} \rho S |\dot{x}|.
$$

For the shape dynamics, we get

$$
\frac{\partial L}{\partial \dot{q}} = B_t^T R^T \ddot{x} + B_r^T \omega + \dot{M} \dot{q}.
$$

Noticing that $\dot{R}^T = -\dot{\omega} R^T$ and $\dot{B}_t = 0$, we have

$$
M \ddot{q} + B_t^T R^T \ddot{x} - B_t^T \dot{\omega} R^T \dot{x} + B_r^T \omega + B_r^T \dot{\omega} - \frac{\partial L}{\partial \dot{q}} = \tau_S,
$$

where we let

$$
\tau_S = [\tau_u - \text{sgn}(\Omega_u) K_{D_u} \Omega_u^2, -\tau_l + \text{sgn}(\Omega_l) K_{D_l} \Omega_l^2, F_a, F_b]^T.
$$

$\tau_u$ and $\tau_l$ are the torques provided by the motors to the upper rotor and lower rotor, respectively. $F_a$ and $F_b$ are the forces also provided by motors controlling the position of the mass block along the $x$ and $y$ directions, respectively. $K_D \Omega^2$ is the drag moment on the rotor, where $K_D = C_D \rho r^5$ and $C_D$ is the dimensionless drag coefficient.

Finally, for the rotational dynamics, we have

$$
\frac{\partial L}{\partial \omega} = K^T R^T \ddot{x} + J \omega + B_r \dot{q},
$$

(5.64)
and
\[
\sum_{i=1}^{3} \frac{\partial L}{\partial R_i} = -(\dot{R}^T \dot{x}) \times \mathcal{K} - (\dot{R}^T \dot{x}) \times \mathcal{B}_i \dot{q} - \sum_{i=1}^{3} \frac{\partial V}{\partial R_i}. \tag{5.65}
\]

After simplification, the rotational equation of motion is
\[
\mathcal{K}^T \dot{R}^T \ddot{x} + J \dot{\omega} + \dot{J} \omega + \mathcal{B}_i \ddot{q} + \dot{\mathcal{B}}_i \dot{q} + \dot{\omega} J \omega + \mathcal{B}_i \dot{q} = \sum_{i=1}^{3} \frac{\partial V}{\partial R_i} + \tau_R, \tag{5.66}
\]

where \(\tau_R = [0, 0, -\text{sgn}(\Omega_u)K_{Du} \Omega_u^2 + \text{sgn}(\Omega_l)K_{Dl} \Omega_l^2]^T\).

Using the Symbolic Toolbox in MATLAB, we have
\[
\frac{\partial L}{\partial q} = \begin{bmatrix}
0, \\
0, \\
-\omega_x \omega_y b + \omega_z \omega h - (\omega_y^2 + \omega_z^2) a + \omega_z \dot{b} - g R_{31}, \\
-m_b (\dot{X}(\omega_y R_{13} - \omega_z R_{12}) + \dot{Y}(\omega_y R_{23} - \omega_z R_{22}) + \dot{Z}(\omega_y R_{33} - \omega_z R_{32})) + \omega_x \omega_y b + \omega_z \omega h - (\omega_y^2 + \omega_z^2) a + \omega_z \dot{b} - g R_{31}, \\
m_b (\dot{X}(\omega_x R_{13} - \omega_z R_{11}) + \dot{Y}(\omega_x R_{23} - \omega_z R_{21}) + \dot{Z}(\omega_x R_{33} - \omega_z R_{31})) - \omega_x \omega_y a - \omega_y \omega h - (\omega_x^2 + \omega_z^2) b + \omega_z \dot{a} - g R_{32}
\end{bmatrix} \tag{5.67}
\]
in equation (5.62) and
\[
\sum_{i=1}^{3} \frac{\partial V}{\partial R_i} = \begin{bmatrix}
m_b g a \\
m_b g b \\
-g (m_u l_1 + m_l l_2 - m_b h)
\end{bmatrix} \tag{5.68}
\]
in equation (5.66).

### 5.4 State Space Equations

In this section, we convert the equations of motion into state space equations suitable for simulation and control system design.
First, we lump equations (5.60), (5.62) and (5.66) into the following format.

\[
\begin{bmatrix}
M & R_K & R_B \\
K^T R^T & J & B_r \\
B_l^T R^T & B_r^T & M
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{\omega} \\
\dot{\bar{q}}
\end{bmatrix}
= \begin{bmatrix}
-R\dot{\omega}K\omega - R\dot{K}\omega - R\dot{\bar{q}}B_r\bar{q} - \frac{\partial V}{\partial x} + \tau_T \\
-\dot{J}\omega - \dot{B}_r\bar{q} - \dot{\omega}J\omega - \dot{\omega}B_r\bar{q} - \sum_{i=1}^{\hat{V}} \frac{\partial V}{\partial R_i} + \tau_R \\
B_l^T \dot{\omega} R^T \bar{x} - B_l^T \omega + \frac{\partial L}{\partial q} + \tau_S
\end{bmatrix}
\]

(5.69)

Let the four control input inputs be

\[u_1 = T_u, u_2 = T_l, u_3 = F_a, u_4 = F_b.\] (5.70)

Let the eighteen state variables be

\[x_1 = \phi, x_2 = \theta, x_3 = \psi, x_4 = \omega_x, x_5 = \omega_y, x_6 = \omega_z,\] (5.71)

\[x_7 = a, x_8 = b, x_9 = \dot{a}, x_{10} = \dot{b}, x_{11} = \Omega_u, x_{12} = \Omega_l\] (5.72)

and

\[x_{13} = x, x_{14} = y, x_{15} = z, x_{16} = \dot{x}, x_{17} = \dot{y}, x_{18} = \dot{z}.\] (5.73)

Then the kinematics equations converted from (5.33) are

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix}
= \begin{bmatrix}
1 & \sin x_1 \tan x_2 & \cos x_1 \tan x_2 \\
0 & \cos x_1 & -\sin x_2 \\
0 & \sin x_1 \sec x_2 & \cos x_1 \sec x_2
\end{bmatrix}
\begin{bmatrix}
x_4 \\
x_5 \\
x_6
\end{bmatrix}.\] (5.74)
and the dynamics equations are

\[
\begin{bmatrix}
\dot{x}_{16} \\
\dot{x}_{17} \\
\dot{x}_{18} \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_{11} \\
\dot{x}_{12} \\
\dot{x}_9 \\
\dot{x}_{10}
\end{bmatrix} = \begin{bmatrix}
\mathcal{M} & R\mathcal{K} & RB_t \\
\mathcal{K}^T R^T & \mathcal{J} & B_r \\
B_i^T R^T & B_r^T & \mathcal{M}
\end{bmatrix}^{-1} \begin{bmatrix}
-R\dot{\omega}\mathcal{K}\omega - R\dot{\mathcal{K}}\omega - R\dot{\omega}B_t\dot{q} - \frac{\partial V}{\partial x_3} + \tau_T \\
-\dot{\mathcal{J}}\omega - \dot{B}_r\dot{q} - \dot{\omega}\mathcal{J}\omega - \omega B_r\dot{q} - \sum_{i=1}^{\mathcal{M}} \frac{\partial V}{\partial R_i} + \tau_R \\
B_i^T \dot{\omega}R^T \dot{x} - B_r^T \dot{\omega} + \frac{\partial L}{\partial q} + \tau_S
\end{bmatrix}
\]  

(5.75)

By definition of the state variables, we also have

\[
\begin{align*}
\dot{x}_7 &= x_9 \\
\dot{x}_8 &= x_{10} \\
\dot{x}_{13} &= x_{16} \\
\dot{x}_{14} &= x_{17} \\
\dot{x}_{15} &= x_{18}
\end{align*}
\]  

(5.76)

Equations (5.74), (5.75) and (5.76) are the state space equations of the system.

### 5.5 Simulation

At first, (5.60), (5.62) and (5.66) are used for simulation in MATLAB and Simulink. However, the existence of algebraic loops is not tolerated by MATLAB. Instead, the previous state space equations (5.74), (5.75) and (5.76) are adopted for simulation. We use the same physical parameters as in Table 4.1.
5.5.1 Equilibrium of the System

Letting the right hand sides of equations (5.74), (5.75) and (5.76) be zeros and considering the physical meaning of hover, we get the equilibrium points of the system corresponding to the hover mode as follows.

\[ x_1^* = 0, x_2^* = 0, x_3^* = 0, x_4^* = 0, x_5^* = 0, x_6^* = 0, \]
\[ x_7^* = 0, x_8^* = 0, x_9^* = 0, x_{10}^* = 0, \]  \hspace{1cm} (5.77)

\[ x_{11}^* = \sqrt{-K_T(m_u l_1 + m_l l_2 - m_t h)g + K_D(m + \sum_{i \in I} m_i)g \over K_T K_D + K_T K_D}, \]  \hspace{1cm} (5.79)

\[ x_{12}^* = \sqrt{K_T(m_u l_1 + m_l l_2 - m_t h)g + K_D(m + \sum_{i \in I} m_i)g \over K_T K_D + K_T K_D}, \]  \hspace{1cm} (5.80)

\[ x_{13}^* = 0, x_{14}^* = 0, x_{15}^* = 0, x_{16}^* = 0, x_{17}^* = 0, x_{18}^* = 0, \]
\[ u_1^* = K_{D_u} x_{11}^2, u_2^* = K_{D_u} x_{12}^2, u_3^* = 0, u_4^* = 0. \]  \hspace{1cm} (5.82)

5.5.2 Numerical Simulation of the System at the Equilibrium

We let the initial conditions for the system to be at the equilibrium. The system responses are plotted in Fig. 5.1

Simulation shows that the equilibrium point computed before is valid, although the motion in the yaw motion tends to diverge due to round-off errors.
5.5.3 Simulation of the System When the Mass Block is Perturbed Slightly

Now we keep the original equilibrium point except adding two identical sine waves having magnitude 0.000001N and frequency $2\pi$ rad/s to the internal forces actuating the mass block. Fig. 5.2 shows that the system is too sensitive to perturbations.
Fig. 5.2. System Response When the Mass Block is Perturbed Slightly
6. ATTITUDE STABILIZATION BASED ON THE SECOND MODEL

The aim of this chapter is to study the linear control of C-fly based on the new model derived in Chapter 5. Same as in Chapter 4, LQR is adopted as the controller to stabilize the attitude.

About the equilibrium points (5.77), (5.78), (5.79), (5.80), (5.81) and (5.82) of the model (5.74), (5.75) and (5.76), let $\Delta x = [x_1 - x_1^*, x_2 - x_2^*, \ldots, x_{12} - x_{12}^*]^T$ and $\Delta u = [u_1 - u_1^*, \ldots, u_4 - u_4^*]^T$, then the linearized model is

$$\dot{\Delta} x = A\Delta x + B\Delta u,$$  \hspace{1cm} (6.1)

where $A$ and $B$ are computed by the ControlDesign Toolbox in Simulink based on the same parameters in the simulation in the previous chapter.

Although only six of the twelve states of the above system are controllable, the system is stabilizable. By the same procedure as in (4.3), (4.4) and (4.5), the controller is designed where we let the weighting matrices be $R = I_{4\times4}$ and $Q = I_{12\times12}$ with the change $Q(1,1) = Q(2,2) = Q(3,3) = 10$ to put more weights on $\phi$, $\theta$ and $\psi$.

The control system is simulated in MATLAB and Simulink and five cases are discussed below.
(a) Upper and Lower Rotor Speeds

(b) The Position of the Mass Block

(c) Tait-Brian Angles of C-fly

(d) Position of C-fly

Fig. 6.1. System response when $(\phi_0, \theta_0, \psi_0) = (17.2^\circ, 17.2^\circ, 17.2^\circ)$

(1) First we let the initial conditions be at the equilibrium point except $(\phi_0, \theta_0, \psi_0) = (17.2^\circ, 17.2^\circ, 17.2^\circ)$. Fig. 6.1 shows that the mass block moves to the front left (the $x$ axis points to the front and the $y$ axis points to the right) and then goes back to the origin in order to drive C-fly back to its hover position. Within 1 second, the attitude is driven to its equilibrium point.
Fig. 6.2. System response when \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\), \(a_0 = 0.1\text{cm}\) and \(b_0 = 0.1\text{cm}\)

(2) Let \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\) and at the same time let the mass block to be off the origin \((a_0 = 0.1\text{cm}\) and \(b_0 = 0.1\text{cm}\)). Fig. 6.2 illustrates that the mass block moves from front right to further front left and then goes back to the origin.
(a) Upper and Lower Rotor Speeds

(b) The Position of the Mass Block

(c) Tait-Brian Angles of C-fly

(d) Position of C-fly

Fig. 6.3. System response when \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\) and the initial value of \((\omega_x, \omega_y, \omega_z) = (1\text{rad/s}, 1\text{rad/s}, 1\text{rad/s})\)

(3) Let \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\) and the initial value of \((\omega_x, \omega_y, \omega_z)\) be \((1\text{rad/s}, 1\text{rad/s}, 1\text{rad/s})\). Fig. 6.3 shows that C-fly first experiences a mild roll motion to the right and then moves back to the equilibrium point but the settling time for the yaw angle is greater than 3 seconds.
Fig. 6.4. System response when \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\), \(\Omega_u = 150\text{rad/s}\) and \(\Omega_l = 300\text{rad/s}\)

(4) According to (5.79) and (5.80), the equilibriant angular speeds are 167.62 rad/s for the upper rotor and 328.73 rad/s for the lower rotor. However, we set the initial values \(\Omega_u = 150\text{rad/s}\) and \(\Omega_l = 300\text{rad/s}\) in addition to \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\). Fig. 6.4 illustrates that \(\Omega_u\) and \(\Omega_l\) are driven up to the equilibrium almost instantly and it takes a large overshoot and long settling time for the yaw angle \(\psi\) which is closely associated to the accelerating motion of the two rotors.
(a) Upper and Lower Rotor Speeds
(b) The Position of the Mass Block
(c) Tait-Brian Angles of C-fly
(d) Position of C-fly

Fig. 6.5. System response when \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\) and the measurements of the three angles are corrupted by zero mean band-limited Gaussian noises with correlation time 0.01 second and covariance 0.005 rad

(5) Lastly, we add zero mean band-limited Guassian noises with correlation time 0.01 second and covariance 0.005 rad to the measurements of \(\phi\), \(\theta\) and \(\psi\) in addition to \((\phi_0, \theta_0, \psi_0) = (5.7^\circ, 5.7^\circ, 5.7^\circ)\). Fig. 6.5 shows that the mass block’s motion on the \(x\) direction has larger average amplitude compared to that of the \(y\) direction. Although the average value of each of \(\phi\), \(\theta\) and \(\psi\) is around the equilibrium, the system is too sensitive to measurement noises.

The observation of the (d)s in Fig. 6.1, 6.2, 6.3, 6.4 and 6.5 reveals that C-fly drifts on the horizontal plane as time elapses. This is because only the attitude is controlled and the translational motion is left to natural response. However, the coupling between the rotational motion and the translational motion make C-fly’s
motion along the vertical direction much weaker than that of the horizontal directions. This is verified by the (d)s mentioned previously.
7. CONCLUSION

This thesis presents a conceptual design of a coaxial double-rotor micro flying robot which control its attitude by shape change. A quasi rigid body model and a multi rigid body system model are derived for the robot, respectively. Linear Quadratic Regulator based on the two models which stabilize the attitude are also studied.

The later model is based on variational principles of mechanics and is more accurate than the fist model whose rotational motion is based on Euler’s equations of motion for a single rigid body. However, the increased accuracy is compromised by the increased complexity which results in the derivation of the model for controller design by hand almost impossible. Nonetheless, software with symbolic computation features such as Matlab, Maple and Mathematica can be employed to do the tedious work.

Furthermore, to avoid getting lost in the complex equations of motion, geometric mechanics and control may be used instead to develop the model. Geometric Mechanics is a modern description of classical mechanics from the perspective of differential geometry. It explores the geometric features of Lagrangian and Hamiltonian mechanics through the tools of vector fields, symplectic geometry, and symmetry [13]. The most significant advantage of geometric mechanics is that it is coordinate free which is inherited from differential geometry. This also make the modeling of the system easier to understand since it is in the most compact form and preserves the geometric meaning compared to the classical way when the geometric feature is lost or hidden. And it completely avoids the singularity problem such as when Euler angles are used.

For simulation and control design, computational geometric and control [14] may be the next new option. Instead of relying on the tradition numerical methods, computational geometric mechanics is the discrete counterpart of geometric mechanics
and thus preserves the geometry. This method is especially useful when doing long-
time simulations when qualitative behavior is more important than the quantitative
results which are often inaccurate.
LIST OF REFERENCES
LIST OF REFERENCES


A. “ANALYTICAL EXPERIMENT” OF A SIMPLE MULTI RIGID BODY SYSTEM

It is stated in [12] that $\tau_R$ is the non-gravitational moment acting on the base body and it is assumed that there is no external moment acting on the auxiliary bodies, while [11] just says $\tau_R$ is the moment acting on the spacecraft (whole system) instead of the base body. Although these two statements seem inconsistent, as far as no external moments act on the auxiliary bodies, it makes no difference to use either one of the two “definitions”.

However, in the case of C-fly, the drag moments do act on the rotor blades which are treated as auxiliary bodies. We want to answer the question “Is $\tau_R$ the external moment on the base body only or the external moment on the whole system?”

We cannot find the answer from [11] since the equations of motion are derived under the hidden assumption that all the external forces are conservative, i.e., derivable from potential energies. There is a small gap between the derivation and the final equations (5.52), (5.53) and (5.54) with nonconservative external force/moment inputs. Hence, the exact nature of the nongravitational external forces in the translation, rotation and shape dynamics equations are vague. Instead of deriving the forced version directly, we present an “analytical experiment” at least to get close to the answer.

Here is the method. First, we try to get an interconnected rigid body system which is as simple as possible. Then we put it in its simplest motion mode in order for rotational motion to appear only in one direction. Therefore, we can use Newton’s method to get the equations of motion and compare them with the Euler-Lagrange-Poincaré
equations to see what is the generalized moment in the equations of rotational motion.

In Fig. A.1, the multibody system is composed of two rigid bodies, the green base body and the yellow auxiliary body (the rotor with the shaft). They have the same axis of axisymmetry. Then we know that if we let the original attitude of the system to be vertical like that in Fig. A.1 and the external nonconservative moment be along the direction of the z-axis, then the system has rotational motion only around the z-axis. Hence, this is an ideal model for our purpose.

The local frame \(oxyz\) is fixed to the base body as shown in the figure and the distance between the center of the auxiliary body and the origin \(o\) of the local frame is \(h\). Hence, the position vector of the auxiliary body in the local frame is \(\rho_A = [0, 0, -h]^T\). Using the same notions as in [12], we have the expression for the position of the auxiliary body in the global frame \(x_A\) as follows.

\[
x_A = x + R\rho_A.
\]

So

\[
\dot{x}_A = \dot{x} + \dot{R}\rho_A + R\dot{\rho}_A = \dot{x} + R\dot{\omega}\rho_A = \dot{x} - R\dot{\omega}_A\omega.
\]

Let the only generalized coordinate \(q_A = \psi_A\), which is the azimuthal angle of the rotor relative to the base body. Then, we have

\[
\omega_A = \omega + [0, 0, 1]^T \dot{\psi}_A.
\]

Substitute the expression for \(\dot{x}_A\) and \(\omega_A\) into that of the kinetic energy \(T\):

\[
T = \frac{1}{2} m_B \dot{x}^T \dot{x} + \frac{1}{2} m_A \dot{x}_A^T \dot{x}_A + \frac{1}{2} \omega^T J_B \omega + \frac{1}{2} \omega_A^T J_A \omega_A,
\]

where the inertia matrix for the base body is

\[
J_B = \begin{bmatrix}
I^B_x & 0 & 0 \\
0 & I^B_y & 0 \\
0 & 0 & I^B_z
\end{bmatrix}
\]
Fig. A.1. A schematic drawing of a simple multibody system

and that of the auxiliary body is

\[ J_A = \begin{bmatrix} I_x^A & 0 & 0 \\ 0 & I_y^A & 0 \\ 0 & 0 & I_z^A \end{bmatrix}, \]

we get

\[ T = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \omega \\ \dot{q}_A \end{bmatrix}^T \begin{bmatrix} M & RK & RB_t \\ K^T R^T & J & B_r \\ B_t^T R^T & B_r^T & M \end{bmatrix} \begin{bmatrix} \dot{x} \\ \omega \\ \dot{q}_A \end{bmatrix} , \]

where

\[ M = \begin{bmatrix} m_B + m_A & 0 & 0 \\ 0 & m_B + m_A & 0 \\ 0 & 0 & m_B + m_A \end{bmatrix}, \]
\[
\mathcal{J} = \begin{bmatrix}
I_x^B + I_x^A + m_A h^2 & 0 & 0 \\
0 & I_y^B + I_y^A + m_A h^2 & 0 \\
0 & 0 & I_z^B + I_z^A
\end{bmatrix},
\]

\[
\mathbb{M} = I_z^A,
\]

\[
\mathcal{K} = \begin{bmatrix}
0 & -m_A h & 0 \\
m_A h & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
\mathcal{B}_r = \begin{bmatrix}
0 \\
0 \\
I_z^A
\end{bmatrix},
\]

and

\[
\mathcal{B}_r = 0.
\]

Since the gravitational potential energy is

\[
V = -m_B g e_3^T x - m_R g e_3^T (x + R \rho_A),
\]

we form the Lagrangian as

\[
L = T - V.
\]

Based on the extended form (v.s. the general form) of the rotational and translational dynamics equations [1]

\[
\mathcal{K}^T R^T \ddot{x} + J \dot{\omega} + \mathcal{B}_r \ddot{q} = -\mathcal{B}_r \dot{q} - \dot{\omega} J \omega - \dot{\omega} B_r \dot{q} - J \omega - \sum_{i=1}^3 \frac{\partial V}{\partial R_i} + [0, 0, \tau_R]^T,
\]

and

\[
\mathcal{B}_r^T R^T \ddot{x} + B_r^T \omega + M \ddot{q} = -B_r^T \dot{\omega} R^T \dot{x} + B_r^T \dot{\omega} R^T \dot{x} - B_r^T \dot{\omega} R^T \dot{x} - \dot{\omega} R^T \omega - M \ddot{q} + \frac{\partial L}{\partial q} + \tau_s,
\]

and noticing that \(\omega = [0, 0, \dot{\psi}]\) and

\[
R = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
we have the rotational equation of motion

\[ I^B_z \ddot{\psi} + I^A_z (\ddot{\psi} + \ddot{\psi}_A) = \tau_R, \]  
(A.1)

and the shape equation of motion

\[ I^A_z (\ddot{\psi} + \ddot{\psi}_A) = \tau_S. \]  
(A.2)

For Newton’s method, we can use the free body diagram analysis.

If we let the torque given by the base body to the auxiliary body be \( T_{BA} \) and the external moment exerted on the auxiliary body be \( T_E \), then for the base body we have

\[ I^B_z \ddot{\psi} = -T_{BA}, \]  
(A.3)

and for the auxiliary body we get

\[ I^A_z (\ddot{\psi} + \ddot{\psi}_A) = T_{BA} + T_E. \]  
(A.4)

Comparing equations A.1 and A.2 with equations A.3 and A.4 we get

\[ \tau_S = T_{BA} + T_E \]

as expected and

\[ \tau_R = T_E, \]

which shows that if external nongravitational moment \( T_E \) acts on the auxiliary body, then in the rotation equation of motion, the generalized moment \( \tau_R \) should include \( T_E \), and in the shape equation of motion, the generalized moment/force \( \tau_S \) is the sum of the moments/forces given by the base body and the external nongravitational environment.