Functional Analysis
Instructor: Professor Thomas Sinclair
Course Number: MA 54600
Credits: Three
Time: 3:30 – 4:20 PM MWF

Description

The course will be divided into two parts. In the first, we will cover the basics of functional analysis: Hilbert Spaces, Linear Operators on Hilbert Space, Banach Spaces, Weak Topologies, and Banach Algebras. This will follow Conway, Chapters I–VII.

In the second part, we will discuss a few applications to probability, harmonic analysis, and ergodic theory. Potential topics, chosen by class interest, are: Kernels and Reproducing Kernel Hilbert Spaces; Random Walks and Harmonic Functions; Concentration of Measure; Gaussian Processes; Grothendieck's Inequality and Semi-Definite Programming; and the Pointwise Ergodic Theorem.

There will be no exams, but regular homework assignments. Grades will be determined by homework.

Required text: Conway, J.B., "A Course in Functional Analysis," Second Ed. *Graduate Texts in Mathematics* **96**, Springer, 1990.

Introduction Abstract Algebra
Instructor: Professor Freydoon Shahidi
Course Number: MA 55300
Credits: Three
Time: 10:30 – 11:20 AM MWF

Description

Pre–requisit: MA 503 or equivalent

Syllabus:

Group Theory: Review of basic definitions and facts including examples of groups: dihedral, symmetric, quaternions; isomorphism theorems, quotient groups, centralizers, normalizers, automorphisms.

Group actions on sets, orbits and stabilizers; representations of a group action as automorphisms of the set and its consequences: conjugation, class formula, p–groups, Sylow's theorem, composition series, solvable and nilpotent groups, simplicity of A_n for $n \geq 5$, direct and semi-direct products.

Ring Theory: Review of basic definitions and facts; isomorphism theorems, ideals, quotient rings, commutative rings: integral domains, maximal and prime ideals, PID, UFD, Euclidean domains, norms, some number theoretic applications: Fermat–Euler, Euler–Gauss theorems on sums of squares; Chinese remainder theorem with applications; ring of quotients for a domain; polynomial rings, reducibility criteria: Gauss's lemma, reduction criteria, Eisenstein polynomials and their irreducibility.

Field Theory: Field extensions, finitely generated and finite extensions, algebraic extensions, generating fields by roots of irreducible polynomials, separability, perfect fields, normal extensions, normal closures, splitting fields, finite fields, primitive elements and simple extensions, algebraically closed fields.

Galois Theory: Galois extensions, fundamental theorem of Galois theory, examples of Galois extensions, roots of unity, cyclotomic extensions, cyclic and abelian extensions, solvable extensions, cyclotomic polynomials, basic facts on Kummer extensions, extensions by radicals, non-solvability of polynomial equations of degree 5 and higher by radicals.

Book: D. Dummit and M. Foote, Abstract Algebra, John Wiley, 3rd Edition.

Lectures: Synchronous Online

Infinite–Dimensional Lie Algebras and Applications

Instructor: Professor Oleksandr Tsymbaliuk
Course Number: MA 59800CIDLA
Credits: Three
Time: 1:30 – 2:45 PM TTh

Description

This course will be a detailed introduction, with proofs, into the structure and representation theory of some of the most important infinite dimensional Lie algebras: Heisenberg algebras, Kac–Moody algebras, and Virasoro algebra.

Major topics to be covered:

- Heisenberg algebra, Virasoro algebra, and affine $\widehat{\mathfrak{g}}$ as universal central extensions
- Representations of Heisenberg algebra, Virasoro algebra, affine $\widehat{\mathfrak{sl}}_n$ via Lie algebras $\mathfrak{gl}_\infty, \mathfrak{a}_\infty$, and application to integrable systems
- Boson–fermion correspondence: vertex operator construction and Schur polynomials

- Feigin–Fuchs–Kac determinant formula for Virasoro and the region of unitarity
- The Sugawara construction and the Goddard–Kent–Olive construction
- Structure and representation theory of Kac–Moody algebras
- The Weyl–Kac character formula and the Kac–Macdonald identities
- Shapovalov–Jantzen–Kac–Kazhdan determinant formula for Kac–Moody algebras

Lectures: Synchronous Online (recorded via Zoom)

References: The material of this course is based on:

(1) Book *"Bombay lectures on highest weight representations of infinite dimensional Lie algebras"* by V. Kac and A. Raina, 2nd edition, 2013.

(2) Expository paper *"Representations of contragredient Lie algebras and the Kac–Macdonald identities"* by B. Feigin and A. Zelevinsky, 1971 (to be distributed in the class).

(3) Book *"Infinite dimensional Lie algebras"* by V. Kac, 1983.

Requirements: To pass the course it will be required to solve homework assignments, which will be assigned every Thursday and due the following Thursday.

Analytic Number Theory: a second course
Instructor: Professor Trevor Wooley
Course Number: MA 59800CNUM
Credits: Three
Time: 10:30 – 11:20 AM MWF

Description

This course serves as a sequel to the first course in analytic number theory taught in Fall 2020. It will serve as a gateway to advanced topics interfacing with problems active in current research concerning the analytic theory of numbers. As such it explores the finer aspects of the distribution of prime numbers in arithmetic progression and consequences for such problems as the Goldbach and Twin Prime problems. Along the way, one encounters the distribution of zeros of the Riemann zeta-function and Dirichlet L-functions, and important estimates for exponential and character sums.

Following Dirichlet's proof in 1837 of the infinitude of primes in arithmetic progressions b modulo q (with b and q coprime), and the proof of the Prime Number Theorem by Hadamard and de la Vallee Poussin in 1896, researchers turned to examine finer questions concerning the distribution of prime numbers. How small is the smallest prime number congruent to b modulo q? What can be said if one averages over the modulus q? How are such results connected with the zeros of Dirichlet L-functions? How close can one come to establishing the Riemann Hypothesis for such functions, and what would this imply about the distribution of primes? Do any of these results lead to interesting implications for the famous conjectures about primes? This is a second course in analytic number theory that explores important methods and estimates that lay the foundation for research in the modern theory. Students interested in broadening their knowledge of analytic methods relevant in harmonic analysis and analytic number theory will find much of this course useful, as will those preparing for research in the area ... and there are many beautiful results and theoretical developments along the way to maintain the interest of enthusiasts. The basic theory of Dirichlet series and the distribution of primes, as described in the Fall course Math 598: A First Course in Analytic Number Theory, will be assumed, though many of the topics of this second course may be studied independently of this material.

Prerequisites: A first course in analytic number theory and basic real and complex analysis.

Assessment: Six or seven (short) problem sets will be offered through the semester, and class participants can demonstrate engagement with the course by any written and/or in-class presentations featuring a reasonable subset of these problems — three levels of difficulty: short problems testing basic skill-sets, extended problems integrating the essential methods of the course, and more challenging problems for enthusiasts with detailed hints available on request.

The course will be based on the instructor's lecture notes, which in turn are based on:

- H. Davenport, Multiplicative Number Theory, 2nd ed., Springer–Verlag, GTM 74, 1980;
- H. Iwaniec and E. Kowalski, Analytic Number Theory, AMS Colloquium Series Vol. 53, 2004;
- H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory, I. Classical Theory, Cambridge Studies in Advanced Mathematics 97, Cambridge University Press, 2007 (and the unpublished volume II).

Basic Topics:

(i) The large sieve for exponential and character sums

(ii) Exponential sums over primes

(iii) The Goldbach problem: sums of two and three primes

(iv) The Bombieri–Vinogradov theorem on primes in arithmetic progression

(v) The Barban–Davenport–Halberstam theorem on the variance of primes in arithmetic progression

(vi) Exponential sum estimates via van der Corput's methods

(vii) The Burgess estimate for character sums in short intervals

(viii) Mean and large values of Dirichlet series

(ix) Approximate functional equations

(x) Bounds for the Riemann zeta function and L-functions in the critical strip

(xi) Exponential sum estimates via Vinogradov's methods

(xii) The zero-free region for the Riemann zeta function and refined asymptotics in the prime number theorem

(xiii) Zero density estimates

(xiv) Primes in short intervals

(xv) The Deuring–Heilbronn phenomenon

(xvi) Linnik's theorem on the smallest prime in an arithmetic progression

Advanced topics depending on demand and available time, may include an introduction to sieves, Maynard's theorem on prime k-tuples, pair correlation and zero density of the zeros of the Riemann zeta function.

Real Algebraic Geometry
Instructor: Professor Saugata Basu
Course Number: MA 59800CRAG
Credits: Three
Time: 11:30 AM – 12:20 PM MWF

Description

Real algebraic geometry concerns the study of algebraic, geometric and topological properties of real algebraic sets (i.e. the real points of varieties defined by real polynomial equations), and more generally of semi-algebraic sets (inequalities are also allowed in the definition). The algebraic part of the theory has to do with properties of ordered rings and fields, the "real spectrum", quadratic forms etc. and was initiated by Hilbert's 17th problem on representing non-negative polynomials by sums of squares of rational functions. The geometric and topological side originates in another one of Hilbert's problems (16th problem) and deals with topological classification of real varieties. The current state of the field is much wider with connections and applications to many different areas of mathematics - including model theory (o-minimality), computational complexity theory, incidence combinatorics etc.

The course will consist of a quick overview of the basics and then concentrating on some problems of current research interest. In lieu of homeworks the students will be asked to read up and present some papers in class.

Texts:

- Real algebraic geometry, Bochnak, Coste, Roy

- Real algebraic varieties, Frdric Mangolte
- Algorithms in real algebraic geometry, 2nd Ed, Basu, Pollack, Roy

Methods of Applied Mathematics I
Instructor: Professor Isaac Harris
Course Number: MA 61100
Credits: Three
Time: 3:00 – 4:15 AM TTh

Description

Banach and Hilbert spaces; Linear Operators; Spectral Theory of Compact Operators; Applications to linear integral equations and regular Sturm-Liouville problems for ODEs.

This course will focus on the application of Functional Analysis. In particular, we will study applications to Integral Equations, (Partial) Differential Equations as well as Inverse Problems.

Assessment: Multiple problem sets will be assigned throughout the semester.

Reference Texts(optional):

1) Introductory Functional Analysis with Applications by Erwin Kreyszig
2) Partial Differential Equations in Action: From Modeling to Theory Sandro Salsa

Finite Element Methods for Partial Differential Equations
Instructor: Professor Zhiqiang Cai
Course Number: MA 61500
Credits: Three
Time: 10:30 – 11:45 AM TTh

Description

The finite element method is the most widely used numerical technique in computational science and engineering. This course covers the basic mathematical theory of the finite element method for partial differential equations (PDEs) including variational formulations of PDEs and construction of continuous finite element spaces. Adaptive finite element method as well as fast iterative solvers such as multigrid and domain decomposition for algebraic systems resulting from discretization will also be presented. The main textbook of this course is the book by Brenner and Scott entitled "The Mathematical Theory of Finite Element Methods".

When time permits, deep neural networks as a new class of approximation functions will also be covered.

Prerequisite: MA/CS 514 or equivalent or consent of instructor.

Topics in Commutative Algebra
Instructor: Professor Linquan Ma
Course Number: MA 69000MA
Credits: Three
Time: 9:30 – 10:20 AM MWF

Description

We will discuss positive characteristic commutative algebra. We will focus on using Frobenius to measure singularities. Some topics include Kunz's theorem, Frobenius splittings, Frobenius structure on local cohomology, connections to birational geometry and MMP, and other applications.

The Cauchy Integral and a PDE Approach to Complex Analysis
Instructor: Professor Steve Bell
Course Number: MA 69300BELL
Credits: Three
Time: 1:30 – 2:20 PM MWF

Description

The only prerequisites for this course are MA 530 and a rudimentary understanding of L^2 as a Hilbert space.

I will cover some developments in complex analysis arising from the remarkable discovery made in 1978 by my mentors, N. Kerzman and E. M. Stein, that the centuries old Cauchy Transform is nearly a self adjoint operator when viewed as an operator on L^2 of the boundary. This new, but fundamental, result represented a shift in the bedrock of complex analysis. It has allowed the classical objects of potential theory and conformal mapping in the plane to be constructed and analyzed in new and very concrete terms.

Another theme of the course will be a PDE approach to complex analysis stemming from the idea of solving the inhomogeneous Cauchy–Riemann equations using the

Cauchy integral formula. We will explore this line of thought in one and several complex variables.

Finally, if time permits and if students are interested, I will prove basic facts about quadrature domains in the complex plane. The unit disc is the simplest example of a quadrature domain because the average value of an analytic function with respect to area measure is the value of the function at the origin. More generally, a quadrature domain has the property that the average value of an analytic function with respect to area measure is given as a fixed finite complex linear combination of the values of the function and its derivatives at finitely many points. I will show how quadrature domains in the plane can be used to simplify the objects of potential theory, and how they can be seen as dense among domains bounded by finitely many Jordan curves. These results will give rise to a "Riemann Mapping Theorem for multiply connected domains" which inspires new methods to view classical objects of potential theory and conformal mapping such as the Bergman kernel, the Szegő kernel, and the Poisson kernel.

Reference book: *The Cauchy transform, potential theory, and conformal mapping*, 2nd edition, Steven R. Bell, CRC Press, 2015.