Theorem 1 (Deligne 1968)

If $f: X \to Y$ is a smooth projective morphism of varieties, then

$$Rf_* \mathbb{Q}_X = \bigoplus R^i f_* \mathbb{Q}[-i]$$

In particular, the Leray spectral sequence degenerates at E_2

$\Uparrow$ + HL

Thm 2 (Deligne's criterion)

If $C \in D^b(A)$, A abelian and $\exists\ l: C \to C[2]$ s.t.

$$l^i: H^{-i}(C) \cong H^i(C) \quad \forall i$$

then $C = \bigoplus H^i(C)[-i]$

(Griffiths–Harris)

Thm 3 (BBDG, 1982)

(Decomposition)

②

If $f: X \to Y$ is projective, Sps

$L \in D^b_c(X, \mathbb{Q})$ is "semisimple of geometric origin" ($\simeq$

$L \cong \bigoplus {}^pH^i(L)[i]$

& direct sums of IC complexes associated to $R^1 f_* \mathbb{Q}$)

Then $\mathbb{R} f_* L$ is also semisimple of geo. orig.

Cor X smooth, then

$$\mathbb{R} f_* \mathbb{Q} \cong \bigoplus IC(L_i)[-i]$$

(3)

Thm (Saito, 1988-90)

if $M \in HM(X)$, $f: X \to Y$ proper, then,

$$(*) \quad f_+ M = \bigoplus_i \underbrace{{}^pH^i(M)[-i]}_{\substack{\text{sum of}\\ \text{Hodge mod.}\\ \text{up to shift}}}$$

Sketch in MHM be compat f_+, also

${}^pH^i f_+ M$ satisfies Hard Lefschetz

$(*)$ follows Deligne's criterion.

Example

Ex (-1) Decomp thm. $\overset{\text{not easy}}{\Rightarrow}$ Deligne's original version.

$\Uparrow$ $\qquad$ $\Uparrow$ easy

Deligne's criterion

Ex 1 $\quad$ Let X singular

$U \subset X$ nonsingular part

$S = X - U$

Choose a res. $f : \tilde{X} \to X$

which is an iso over U.

$$\mathbb{R} f_* \mathbb{Q} = \bigoplus \underbrace{IC(L_i)}_{\substack{\text{minimal extension of} \\ \text{a local system} \\ \text{as a perverse sheaf}}}[n_i]$$

Decomp.

Use geometry $(n = \dim X)$

on U $f = \mathrm{id}$.

$$\mathbb{R} f_* \mathbb{Q}^{[n]} = \underbrace{\mathbb{Q}[n]|_U}$$

$$\Rightarrow \quad \mathbb{R} f_* \mathbb{Q} = IC(\mathbb{Q}) \oplus \left(\begin{array}{c}\text{sheaf} \\ \text{supp.} \\ \text{on } S\end{array}\right)$$

Cor

$$H^i(\tilde{X}, \mathbb{Q}) = IH^i(X, \mathbb{Q}) \oplus \boxed{?}$$

Then $IH^i(X, \mathbb{Q})$ is a sub Hodge structure of $H^i(\tilde{X}, \mathbb{Q})$

$$\downarrow$$

$$\cancel{IH^i(X, \mathbb{Q}) = \ker\left[H^i(\tilde{X}, \mathbb{Q}) \to H^i(E, \mathbb{Q})\right]} \quad \cancel{H^i(E, \mathbb{Q})}$$

(2) Let $f : \hat{X} \to X$ be a blow up of a smooth variety along a smooth $Z \subset X$ of codim k.

$$\left\{ \begin{array}{l} E = f^{-1}(Z) = \mathbb{P}^{k-1}\text{-bundle on } Z \\ \downarrow g \\ Z. \end{array} \right.$$

$$\mathbb{R} f_* \mathbb{Q} = \underline{\mathbb{Q}_X} \oplus \left(\underbrace{\text{stuff on } Z}_{\text{Restrict to } Z} \right)$$

$$\mathbb{Q}_Z \oplus \mathbb{Q}_Z[2] \oplus \cdots = \mathbb{Q}_Z[2k-2]$$

$$\mathbb{Q}_X \oplus \underbrace{\mathbb{Q}_Z[2] \oplus \cdots \mathbb{Q}_Z[2(k-2)]}_{\text{as Hodge modules need a "Tate twist"}}$$

(3) (Göttsche – Soergel) 7

Let X be a (smooth) alg surface.

$$X^{(n)} = S^n X = \underbrace{X \times \cdots \times X}_{n} / S_n$$

$X^{[n]}$ = Hilbert scheme of 0-dim'l subschemes of X of length n.

$$f : X^{[n]} \longrightarrow X^{(n)}$$

(Fogarty late 60's)

f is a resolution of singularities.

$$Rf_* \mathbb{Q} = \mathbb{Q}_{X^{(n)}} \oplus \left(\begin{matrix}\text{stuff} \\ \text{on singularities}\end{matrix}\right)$$

Singular set of $X^{(n)}$ are diagonals

$\Delta_I = \{(x_1 \ldots x_n) \mid x_i \text{ equal for } i \in P \in I\}$

where I is a partition of $\{1, \ldots n\}$.

8

$$R f_* \mathbb{Q} = \mathbb{Q}_{X(n)} \oplus \cancel{\oplus} \underbrace{\mathbb{Q}_{\Delta_I}^{?}[?]}_{\text{pure Tate.}}$$

---

(4) $\underbrace{f : X}_{\text{smooth}} \longrightarrow \underbrace{C}_{\text{smooth curve}}$ onto, projective with connected fibres.

Let $S \subset C$ be the discriminant

$U = C - S \overset{j}{\hookrightarrow} C$ f smooth & projective

$$R f_* \mathbb{Q} = \ ?$$

Apply Deligne to $f|_{f^{-1}U}$ (9)

$$Rf_* \mathbb{Q}|_U = \bigoplus R^i f_* \mathbb{Q}|_U [-i]$$

(*) $$\boxed{Rf_* \mathbb{Q}} = \bigoplus \underbrace{IC^{\cdot}(R^i f_* \mathbb{Q})}_{j_{!*} R^i f_* \mathbb{Q}}^{[-i]} \oplus \left(\begin{array}{c}\text{stuff} \\ \text{on} \\ S\end{array}\right)$$

Application local invariant cycle thm (Clemens–Schmid early 1970's)

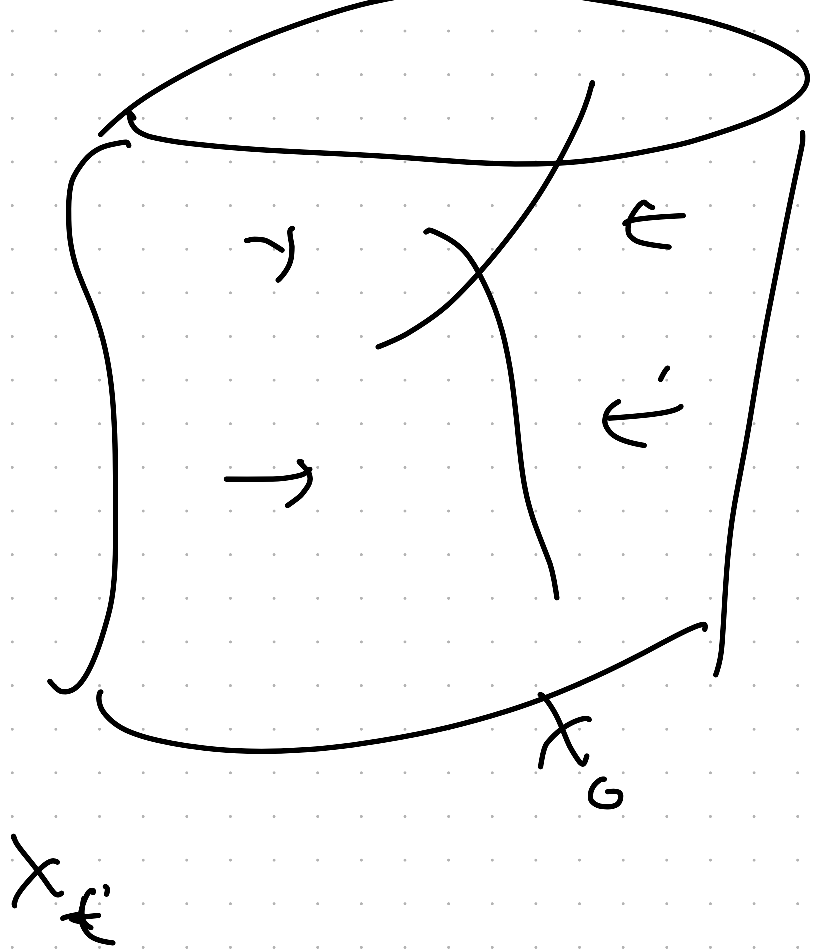
$\Delta \subset \mathbb{C}$

center at a critical pt 0.

Replace X by $f^{-1}\Delta$ etc.

$$H^i(X_0) \cong H^i(X) \xrightarrow{\text{onto}} H^i(X_t)^{T-1}$$

(follows from decomposition thm)



$$H^i(X_0) \twoheadrightarrow H^i(X_\epsilon)$$

onto

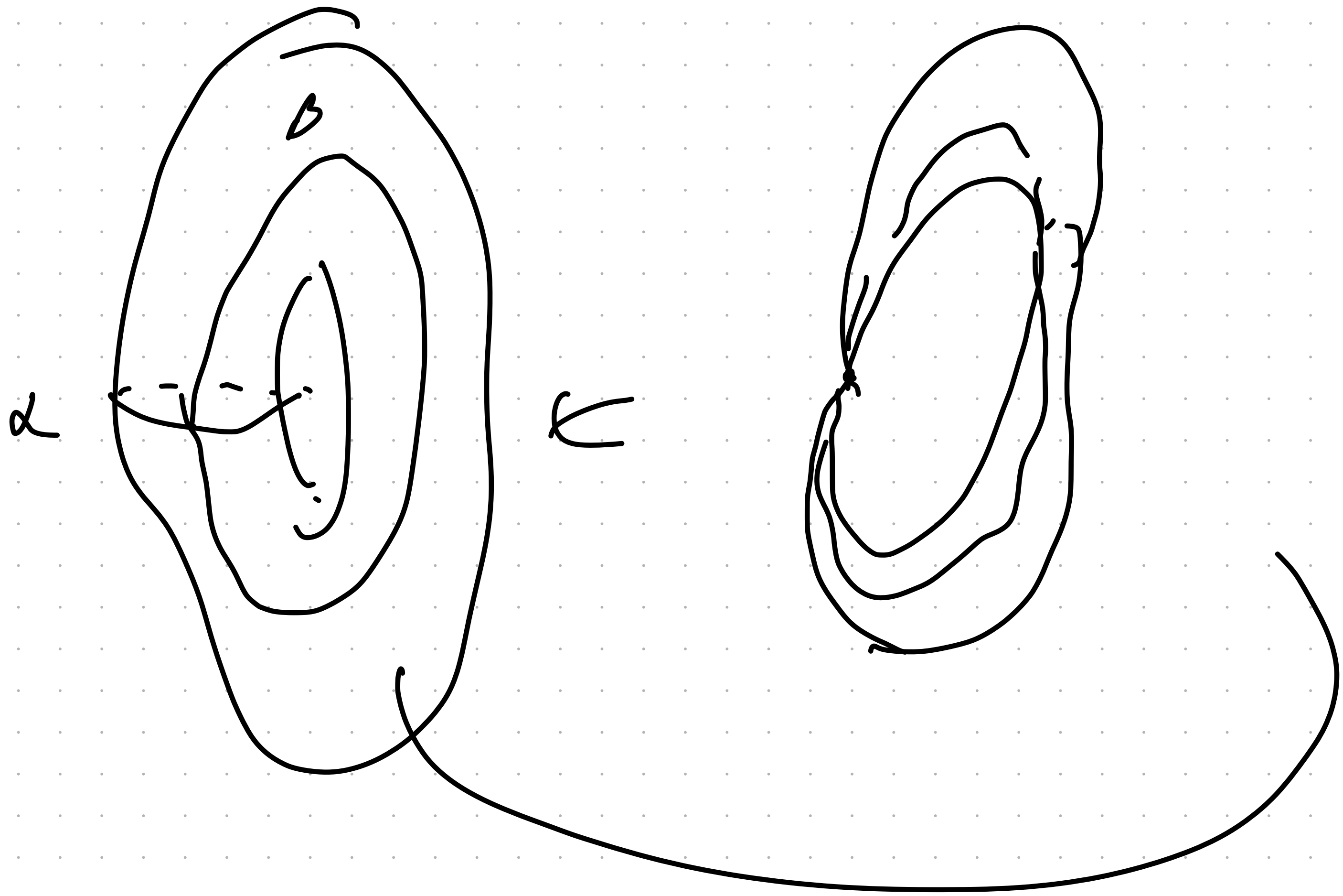
---

with the same pf can work
C by higher dim base.

---

de Cataldo - Migliorini have
a 3rd pf of decomp

Bull AMS.



$\alpha \to \alpha$

$\beta \to \beta\, k\, \alpha$

(5) $(D . A)$ $\Leftarrow$ decoupl fn.

$$\langle \alpha_1 \dots \alpha_{2g}, \beta \mid [\alpha_1, \alpha_2] \cdots [\alpha_{2g-1}, \alpha_{2g}] = \cancel{\beta} \rangle$$

is not Kähler