1. What is the aim of the theory?

Suppose $X \to$ complex analytic variety

$$\begin{array}{l} X \\ \downarrow f \quad \text{a proper morphism which is smooth over } D^* = D \setminus \{0\} \\ D \leftarrow \text{unit disc} \end{array}$$

Qn. Then find the relation between $H^*(X_t)$ & $H^*(X_0)$ for $t \neq 0$.

We know the answer to this in some cases. For ex. if f was smooth & proper over D then $R^i f_* \mathbb{Z}$ are local systems on D (smooth & proper base change), thus $H^i(X_t) \xrightarrow{\sim} H^i(X_0)$.

More specifically we want to understand

a) Relation between the stalks $(R^i f_* \mathbb{Z})_{t \neq 0}$ & $(R^i f_* \mathbb{Z})_0$

b) the monodromy action on $(R^i f_* \mathbb{Z})_t$.

2. Sheaves on a Disk:

Let X be a complex analytic variety. Let $x \in X$ be a base point. Then we have an equivalence of the following categories.

$$\left\{ \begin{array}{c} \text{finite dimensional complex} \\ \text{Representations of } \pi_1(X, x) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{complex} \\ \text{Local systems } \mathcal{L} \text{ on } X \end{array} \right\}$$

Moreover under this equivalence $H^0(X,\mathcal{F}) = \mathcal{F}_x^{\pi_1(X,x)}$

Sheaves on a disc:

Prop^n: The following categories are equivalent

$$\left\{\begin{array}{l}\text{sheaves on } \Delta \\ \text{whose restriction} \\ \text{to } \Delta^* \text{ is a local} \\ \text{system}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{triple } (\mathcal{F}_0, \mathcal{F}_t, \alpha) \\ \text{consisting of } \mathbb{C}[\pi_1(\Delta^*)]\text{-vector} \\ \text{spaces } \mathcal{F}_0, \mathcal{F}_t \text{ with the action being} \\ \text{trivial on } \mathcal{F}_0 \text{ \& an equivariant} \\ \text{map } \alpha: \mathcal{F}_0 \to \mathcal{F}_t\end{array}\right\}$$

Pf Given a sheaf $\mathcal{F}$ on Δ whose restriction to Δ^* is a local system

we set $\mathcal{F}_0 \Rightarrow$ Stalk of $\mathcal{F}$ at 0

$\mathcal{F}_t \Rightarrow$ Stalk of $\mathcal{F}$ at $t \neq 0$

The map α: given any $u \in \mathcal{F}_0$, choose an open nbd. $U \ni 0$ s.t $u \in H^0(U, \mathcal{F})$

We define $\alpha(u) = \operatorname{Im}(u)$ in $H^0(U^*, \mathcal{F}) = \mathcal{F}^{\pi_1(\Delta^*)}$

Easy to see that this ind. of the choice of U.

Conversely given $(\mathcal{F}_0, \mathcal{F}_t, \alpha)$, first get a local system on Δ^* using $\mathcal{F}_t$. Then glue this with the stalk at 0 to get a sheaf on Δ.

2.1 Now we realize the triple $(\mathcal{F}_0, \mathcal{F}_t, \alpha)$ geometrically.

Let $\tilde{D}^*$ be an universal covering space of D^*.

Define $\tilde{D} := \tilde{D}^* \amalg \{0\}$ & give a topology such that

a) $\tilde{D}^* \underset{\text{open}}{\hookrightarrow} \tilde{D}$

b) basis of open nbd.s of $0 \in \tilde{D}$ are of the form $p^{-1}(U)$, U an open nbd. of $0 \in D$.

Then we may extend $p' : \tilde{D}^* \to D^*$ to a continuous map $p : \tilde{D} \to D$. Thus have a comm. diagram

$$\begin{tikzcd}
\{0\} \arrow[r, hook, "\bar{i}"'] \arrow[d, equal] & \tilde{D} \arrow[d, "p"] & \tilde{D}^* \arrow[l, hook', "\bar{j}"'] \arrow[d, "p'"] \\
\{0\} \arrow[r, hook, "i"'] & D & D^* \ni t \arrow[l, hook', "j"']
\end{tikzcd}$$

Let $\mathcal{F}$ be a sheaf on D. Then we have a natural map.

$$p^*\mathcal{F} \longrightarrow \bar{j}_* \bar{j}^* p^* \mathcal{F} = \bar{j}_* p'^*(\mathcal{F}|_{D^*})$$

pulling this back along $\bar{i}$ gives a map of sheaves on $\{0\}$

$$\begin{array}{c} \bar{i}^* p^* \mathcal{F} \\ \| \\ \mathcal{F}_0 \end{array} \longrightarrow \bar{i}^* \bar{j}_* (\bar{\mathcal{F}}|_{D^*}) = \mathcal{F}_t$$

naturally $\pi_1(D^*)$ equivariant.

The upshot is $\mathcal{F}_0, \mathcal{F}_t$ naturally live on $\{0\}$ & there is an equivariant map $\alpha : \mathcal{F}_0 \to \mathcal{F}_t$.

3. Nearby cycles and Vanishing Cycles:

Given any space Y we define two abelian categories.

a) $Sh(Y \times \mathcal{D}) \to$ triple $(\mathcal{F}_0, \mathcal{F}_\eta, \alpha)$

Sheaves on Y with an action of $\pi_1(D^*)$

$\alpha : \mathcal{F}_0 \to \mathcal{F}_\eta$ an equivariant map.

b) $Sh(Y \times \mathcal{D}^*) \to$ a sheaf $\mathcal{G}$ on Y with an action of $\pi_1(D^*)$.

Rmk.: The notation $Y \times \mathcal{D}$ & $Y \times \mathcal{D}^*$ have their origins in a suitable product topos.

We also have the following functors.

$$Sh(Y \times \mathcal{D}) \underset{sp^*}{\overset{j^*}{\rightrightarrows}} Sh(Y \times \mathcal{D}^*)$$

$$j^*(\mathcal{F}_0, \mathcal{F}_\eta, \alpha) \longrightarrow \mathcal{F}_\eta$$

$$sp^*(\mathcal{F}_0, \mathcal{F}_\eta, \alpha) \longrightarrow \mathcal{F}_0$$

We can upgrade this entire picture at the level of derived categories. Thus we have $D^*(Y \times \mathcal{D}, \Lambda)$ & the derived $D^*(Y \times D^*, \Lambda)$ versions of sp^* & j^*.

Hence beginning with any $\mathcal{F} \in D(Y \times \mathcal{D}, \Lambda)$ we get objects.
$sp^*\mathcal{F}$ & $j^*\mathcal{F} \in D(Y \times \mathcal{D}^*, \Lambda)$ & a morphism.

$$\boxed{sp^*\mathcal{F} \xrightarrow[\alpha]{} j^*\mathcal{F}.}$$

$\Phi(\mathcal{F})$ & Var.:

The vanishing cycle complex of $\mathcal{F}$ is the cone

$$sp^*\mathcal{F} \to j^*\mathcal{F} \to \Phi(\mathcal{F}) \xrightarrow{+1} \quad \text{in } D(Y \times \mathcal{D}^*, \Lambda).$$

Rmk.: a) $\Phi(\mathcal{F})$ as constructed above is not unique. For a more precise construction see SGA 7, II, Exp XIII Section 1.4.

b) Note that $\Phi(\mathcal{F})$ by construction has an action of $\pi_1(\mathcal{D}^*)$ & there is a long exact sequence in cohomology

$$\to \mathbb{H}^i(Y, sp^*\mathcal{F}) \to \mathbb{H}^i(Y, j^*\mathcal{F}) \to \mathbb{H}^i(Y, \Phi(\mathcal{F})) \to \cdots$$

equivariant for the action of $\pi_1(\mathcal{D}^*)$.

Now let $T \in \pi_1(\mathcal{D}^*)$ be the oriented generator. We define $\mathrm{Var}(T)$ to be the induced map $\Phi(\mathcal{F}) \to j^*\mathcal{F}$ coming from.

$$\begin{tikzcd}
sp^*\mathcal{F} \arrow[r] \arrow[d, "T-1=0"] & j^*\mathcal{F} \arrow[r] \arrow[d, "T-1"] & \Phi(\mathcal{F}) \arrow[r] \arrow[d, "T-1"] \arrow[dl, dotted, "\mathrm{Var}(T)"'] & sp^*\mathcal{F}[1] \arrow[d, "0"] \\
sp^*\mathcal{F} \arrow[r] & j^*\mathcal{F} \arrow[r] & \Phi(\mathcal{F}) \arrow[r] & sp^*\mathcal{F}[1]
\end{tikzcd}$$

Having discussed an abstract set-up let us give concrete examples of $Sh(Y \times \tilde{D})$

3.1: The case of a morphism:

Suppose $\begin{matrix} X \\ \downarrow f \\ D \end{matrix}$ a morphism. we want to define the nearby cycles & vanishing cycles functor

$$R\Psi : D^+(X, \Lambda) \longrightarrow D^+(X_0 \times \tilde{D}, \Lambda)$$
$$R\Phi : D^+(X, \Lambda) \longrightarrow D^+(X_0 \times \tilde{D}^*, \Lambda)$$

We have a diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "\bar{i}"] \arrow[d, equal] & \bar{X} \arrow[d, "p"] & \bar{X}^* \arrow[l, hook', "\bar{j}"'] \arrow[d, "p'"] \\
X_0 \arrow[r, hook, "i"'] & X & X^* \arrow[l, hook', "j"]
\end{tikzcd}$$

over

$$\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[d, equal] & \tilde{D} \arrow[d] & \tilde{D}^* \arrow[l, hook'] \arrow[d] \\
\{0\} \arrow[r, hook] & D & D^* \arrow[l, hook']
\end{tikzcd}$$

Define $R\Psi(\mathcal{F}) = \left(i^*\mathcal{F},\ i^*\bar{j}_*\bar{j}^*p^*\mathcal{F},\ i^*\left(p^*\mathcal{F} \to \bar{j}_*\bar{j}^*p^*\mathcal{F}\right) \right)$

Thus

$$sp^* R\Psi(\mathcal{F}) = i^*\mathcal{F}.$$

$$j^* R\Psi(\mathcal{F}) = i^*\bar{j}_* p'^*\mathcal{F}.$$

Also written as $R\Psi_{\bar{\eta}}(\mathcal{F})$

Thus we have a long exact sequence of cohomology gps on X_0

$$\to \mathbb{H}^i(X_0, \mathcal{F}_0) \longrightarrow \mathbb{H}^i(X_0, R\Psi_{\bar{\eta}}(\mathcal{F})) \to \mathbb{H}^i(X_0, R\Phi(\mathcal{F})) \to \cdots$$

Lemma: Suppose f is proper. Then $\mathbb{H}^i(X_0, R\Psi_{\bar{\eta}}(\Lambda)) \xrightarrow{\sim} \mathbb{H}^i(X_t, \Lambda)$.

Pf. A consequence of the proper base change thm.

Thus when f is proper we have a simpler long exact sequence.

$$\to H^i(X_s, \Lambda) \to H^i(X_t, \Lambda) \to \mathbb{H}^i(X_0, R\Phi(\Lambda)) \to$$

Thus justifying the name vanishing cycles.

Question: How to think about $R\Psi_{\bar{\eta}}\Lambda$?

Ans.
Suppose $\mathcal{F} = \Lambda$, then to understand the change in cohomology

one needs to understand $R\psi_{\bar\eta}\Lambda$. For any point $x \in X_0$. The
$$R\psi_{\bar\eta}\Lambda = \bar i^* \bar j_* \Lambda.$$
stalks of the q^{th}

q^{th} Cohomology $\left(\bar i^* R^q \bar j_* \Lambda\right)_{x_0}$ are canonically the q^{th} cohomology of a Milnor Ball.

In fact if f is smooth at x_0 then $\left(R^i\psi_{\bar\eta}\Lambda\right)_{x_0} = 0$ for $i \neq 0$ & $i=0$ it is Λ.

4. Relationship with de Rham Cohomology:

Now we summarize the results in SGA 7, II, Exp. XIV, Section 4.

As before we have a diagram.

$$\begin{tikzcd}
X_0 \arrow[r, hook, "\bar i"] \arrow[d, equal] & \bar X \arrow[d, "p"] & \bar X^* \arrow[l, hook', "\bar j"'] \arrow[d, "p'"] \\
X_0 \arrow[r, hook, "i"] & X & X^* \arrow[l, hook', "j"']
\end{tikzcd}$$

Suppose $\bar X^*$ is smooth. Then.

<u>Propⁿ</u>: $R\psi_{\bar\eta}(\mathbb{C}) \simeq \bar i^* \bar j_* \Omega^*_{\bar X^*}$ in $D^b(X_0, \mathbb{C})$.

<u>Pf</u>. On $\bar X^*$ by the holo. Poincare Lemma we have an iso.

$\bar{\mathbb{C}} \xrightarrow{\sim} \Omega^*_{\bar X^*}$. This give a morphism.

$\bar j_* \Omega^*_{\bar X^*} \longrightarrow R\bar j_* \mathbb{C}$. $\leftarrow$ This would trivially have been an iso. if we had 'R' on the right.

Now. we claim.

$$\bar{j}_* \Omega^*_{\bar{X}^*} \xrightarrow{\sim} R\bar{j}_* \mathbb{C}.$$ This is trivially true on $\bar{X}^*$.

To check this on X_0. We need to show that $\left(R^q \bar{j}_* \Omega^*_{\bar{X}^*}\right)_x = 0 \quad \forall x \in X_0$.

Or equivalently that any $x \in X_0$ has a basis of stein neighbourhoods.

This follows from the fact that $X^* \hookrightarrow X$ is defined by the complement of one equation.

Thus for any local system V on X^*.

$$\boxed{\bar{i}^* \bar{j}_* \Omega^*_{\bar{X}^*}(p'^* V) \xrightarrow{\sim} R\Psi_\eta V.}$$

In fact more is true. There exists a subcomplex of $\bar{i}^* \bar{j}_* \Omega^*_{\bar{X}^*}(p'^* V)$ for which this iso. holds.

4.1 quasi-unipotent sections with finite determination:

Let $\mathcal{F}$ be a sheaf on X^*. Let $\bar{\mathcal{F}}$ be its image in $\bar{X}^*$. An element $f \in H^0(\bar{X}^*, \bar{\mathcal{F}})$ is said to be of finite determination if

the span of $\langle T^n f, n \in \mathbb{N} \rangle$ is a finite dimensional space — T: generator of $\pi_1(D^*)$

$$\Updownarrow$$

There exists a polynomial $P(T)$ such that $P(T) f = 0$.

Moreover f is said to be quasi-unipotent if $P(T) = (T^m - 1)^N$.

For any locally free sheaf $\mathcal{F}$ on X^* obtained as a restriction of sheaf on X we can also talk of sections (of $\mathcal{F}$) with moderate growth along X_0.

Thus in particular we have.

$\Omega^{mqu}_{\eta}(\Omega^*_X) \leftarrow$ Sections of $i^* \bar{j}_* \Omega^*_{\bar{X}}$ which are images of sections of $\bar{j}_* \Omega^*_{\bar{X}}$ with moderate growth & quasi-unipotent finite determinants. Then.

Thm. $\Psi^{mqu}_{\eta}(\Omega^*_X) \xrightarrow{\sim} R\Psi_{\eta}\mathbb{C}.$