Part 2

Hodge Theory

CHAPTER 7

The Hodge theorem for Riemannian manifolds

Thus far, our approach has been pretty much algebraic or topological. We are going to need a basic analytic result, namely the Hodge theorem which says that every de Rham cohomology class has a unique "smallest" element. Standard acccounts of basic Hodge theory can be found in the books of Griffiths-Harris [**GH**], Warner [**Wa**] and Wells [**W**]. However, we will depart slightly from these treatments by outling the heat equation method of Milgram and Rosenbloom [**MR**]. This is an elegant and comparatively elementary approach to the Hodge theorem. As a warm up, we will do a combinatorial version which requires nothing more than linear algebra.

7.1. Hodge theory on a simplicial complex

In order to motivate the general Hodge theorem, we work this out for a finite simplicial complex. Let $K = (V, \Sigma)$ be a finite simplicial complex. Choose inner products on the spaces of cochains $C^*(K, \mathbb{R})$. For each simplex S, let

$$\delta_S(S') = \begin{cases} 1 \text{ if } S = S' \\ 0 \text{ otherwise} \end{cases}$$

These form a basis. A particularly natural choice of inner product is determined by making this basis orthonormal. Let $\partial^* : C^i(K, \mathbb{R}) \to C^{i-1}(K, \mathbb{R})$ be the adjoint to ∂, and let $\Delta = \partial\partial^* + \partial^*\partial$. Δ is the discrete Laplacian.

Lemma 7.1.1. *Let α be a cochain. The following are equivalent:*

1. $\alpha \in (im\partial)^\perp \cap ker\partial$.
2. $\partial\alpha = \partial^*\alpha = 0$.
3. $\Delta\alpha = 0$.

Proof. We prove the equivalences of (1) and (2), and (2) and (3). Suppose that $\alpha \in (im\partial)^\perp \cap ker\partial$. Then of course $\partial\alpha = 0$. Furthermore

$$||\partial^*\alpha||^2 = \langle \alpha, \partial\partial^*\alpha \rangle = 0$$

implies that $\partial^*\alpha = 0$. Conversely, assuming (2). $\alpha \in ker\partial$, and

$$\langle \alpha, \partial\beta \rangle = \langle \partial^*\alpha, \beta \rangle = 0.$$

If $\partial\alpha = \partial^*\alpha = 0$, then $\Delta\alpha = 0$. Finally suppose that $\Delta\alpha = 0$. Then

$$\langle \Delta\alpha, \alpha \rangle = ||\partial\alpha||^2 + ||\partial^*\alpha||^2 = 0$$

which implies (2).

□

The cochains satisfying the above conditions are called harmonic.

Lemma 7.1.2. *Every simplical cohomology class has a unique harmonic representative. These are precisely the elements of smallest norm.*

Proof. To prove the first statement, we check that the map

$$h : (im\partial)^\perp \cap ker\partial \to H^n(K, \mathbb{R})$$

sending a harmonic cochain to its cohomology class is an isomorphism. If α lies in the kernel of h then $\alpha = \partial\beta$ which implies that

$$||\alpha||^2 = \langle \alpha, \partial\beta \rangle = 0$$

Given a cochain α with $\partial\alpha = 0$. We can decompose it as $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in (im\partial)^\perp \cap ker\partial$ and $\alpha_2 \in im\partial$. Thus the cohomology class of $\alpha = h(\alpha_1)$.

Given a harmonic element α,

$$||\alpha + \partial\beta||^2 = ||\alpha||^2 + ||\partial\beta||^2 > ||\alpha||^2$$

unless $\partial\beta = 0$. □

Exercises

1. Prove that the space of cochains can be decomposed into an orthogonal direct sum
$$C^n(K) = ker\Delta \oplus im\partial \oplus im\partial^*$$
2. Prove that Δ is a positive semidefinite symmetric operator.
3. Use 2 to prove that limit of the "heat kernel"
$$H = \lim_{t\to\infty} e^{-t\Delta}$$
exists and is the orthogonal projection to $ker\Delta$.
4. Let G be the endomorphism of $C^n(K)$ which acts by 0 on $ker\Delta$ and by $\frac{1}{\lambda}$ on the λ-eigenspaces of Δ with $\lambda \neq 0$. Check that $I = H + \Delta G$.

7.2. Harmonic forms

Let X be an n dimensional compact oriented manifold. We want to prove an analogue of lemma 7.1.2 for de Rham cohomology. In order to formulate this, we need inner products. A Riemannian metric $(,)$, is a family of inner products on the tangent spaces which vary in a C^∞ fashion. This means that inner products are determined by a tensor $g \in \Gamma(X, \mathcal{E}^1_X \otimes \mathcal{E}^1_X)$. The existence of Riemannian metrics can be proved using a standard partition of unity argument [Wa]. A metric determines inner products on exterior powers of the cotangent bundle which will also be denoted by $(,)$. In particular, the inner product on the top exterior power gives a tensor $\det(g) \in \Gamma(X, \mathcal{E}^n_X \otimes \mathcal{E}^n_X)$. Since X is oriented it is possible to choose a positive square root of $\det(g)$ called the volume form $dvol \in \mathcal{E}^n(X)$. The Hodge star operator is a $C^\infty(X)$-linear operator $* : \mathcal{E}^p(X) \to \mathcal{E}^{n-p}(X)$, determined by

$$\alpha \wedge *\beta = (\alpha, \beta) dvol.$$

One can choose a local orthonormal basis or frame e_i for $\mathcal{E}^1_X$ in a neighbourhood of any point. Then $*$ is easy to calculate in this frame, e.g. $*e_1 \wedge \ldots e_k = e_{k+1} \wedge \ldots e_n$. From this one can check that $** = \pm 1$. The spaces $\mathcal{E}^p(X)$ carry inner products:

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) dvol = \int_X \alpha \wedge *\beta$$

and hence norms.

Then basic result of Hodge (and Weyl) is:

THEOREM 7.2.1 (the Hodge theorem). *Every de Rham cohomology class has a unique representative which minimizes norm. This is called the harmonic representative.*

As an application of this theorem, we have get a new proof of a Poincaré duality in strengthened form.

COROLLARY 7.2.2. *(Poincaré duality, version 2). The pairing*

$$H^i(X, \mathbb{R}) \times H^{n-i}(X, \mathbb{R}) \to \mathbb{R}$$

induced by $(\alpha, \beta) \mapsto \int \alpha \wedge \beta$ *is a perfect pairing*

PROOF. $*$ induces an isomorphism between the space of harmonic i-forms and $n-i$-forms. This proves directly that $H^i(X, \mathbb{R})$ and $H^{n-i}(X, \mathbb{R})$ are isomorphic.

Consider the map

$$\lambda : H^i(X, \mathbb{R}) \to H^{n-i}(X, \mathbb{R})^*$$

given by $\lambda(\alpha) = \beta \mapsto \int \alpha \wedge \beta$. We need to prove that λ is an isomorphism. Since these spaces have same dimension, it is enough to prove that $ker(\lambda) = 0$. But this clear since $\lambda(\alpha)(*\alpha) \neq 0$ whenver α is a nonzero harmonic form. □

To understand the meaning of the harmonicity condition, we will find the Euler-Lagrange equation. Let α be a harmonic p-form. Then for any for any $(p-1)$-form β, we would have to have

$$\frac{d}{dt}||\alpha + td\beta||^2|_{t=0} = 2\langle \alpha, d\beta \rangle = 2\langle d^*\alpha, \beta \rangle = 0,$$

which forces $d^*\alpha = 0$, where d^* is the adjoint of d. A straight forward integration by parts shows that $d^* = \pm * d*$. Thus harmonicity can be expressed as a pair of differential equations $d\alpha = 0$ and $d^*\alpha = 0$. It is sometimes more convenient to combine these into a single equation. For this we need

DEFINITION 7.2.3. *The Hodge Laplacian is* $\Delta = d^*d + dd^*$.

Then from the identity $\langle \Delta\alpha, \alpha \rangle = ||d\alpha||^2 + ||d^*\alpha||^2$, we conclude

LEMMA 7.2.4. *The following are equivalent*

1. α *is harmonic.*
2. $d\alpha = 0$ *and* $d^*\alpha = 0$.
3. $\Delta\alpha = 0$.

The hard work is contained in the following result, whose proof will be postoned until the next section.

THEOREM 7.2.5. *There are linear operators* H *(harmonic projection) and* G *(Green's operator) taking* C^∞ *forms to* C^∞ *forms, which are characterized by the following properties*

1. $H(\alpha)$ *is harmonic,*
2. $G(\alpha)$ *is orthogonal to the space of harmonic forms,*
3. $\alpha = H(\alpha) + \Delta G(\alpha)$,

for any C^∞ *form* α.

COROLLARY 7.2.6. *There is an orthogonal direct sum*

$$\mathcal{E}^i(X) = (\textit{harmonic forms}) \oplus d\mathcal{E}^{i-1}(X) \oplus d^*\mathcal{E}^{i+1}(X)$$

PROOF OF THEOREM 7.2.1. We prove the existence part of the theorem. The uniqueness is straightforward and left as an exercise. Any α can be written as $\alpha = \beta + dd^*\gamma + d^*d\gamma$ with β harmonic. All these terms are in fact orthogonal to each other, so

$$||d^*d\gamma||^2 = \langle d^*d\gamma, \alpha\rangle = \langle d\gamma, d\alpha\rangle,$$

and this vanishes if α is closed. Therefore α is cohomologous to the harmonic form β. □

Before doing the general case, let us work out the easy, but instructive, example of the torus

EXAMPLE 7.2.7. $X = \mathbb{R}^n/\mathbb{Z}^n$, *with the Euclidean metric. A differential form α can be expanded in a Fourier series*

$$\alpha = \sum_{\lambda\in\mathbb{Z}^n}\sum_{i_1<\ldots<i_p} a_{\lambda,i_1\ldots i_p} e^{2\pi i\lambda\cdot\mathbf{x}} dx_{i_1}\wedge\ldots\wedge dx_{i_p} \tag{17}$$

By direct calculation, one finds the Laplacian

$$\Delta = -\sum \frac{\partial^2}{\partial x_i^2},$$

the harmonic projection

$$H(\alpha) = \sum a_{0,i_1\ldots i_p} dx_{i_1}\wedge\ldots\wedge dx_{i_p}$$

and Green's operator

$$G(\alpha) = \sum_{\lambda\in\mathbb{Z}^n-\{0\}}\sum \frac{a_{\lambda,i_1\ldots i_p}}{4\pi^2|\lambda|^2} e^{2\pi i\lambda\cdot\mathbf{x}} dx_{i_1}\wedge\ldots\wedge dx_{i_p}.$$

Since the image of H are forms with constant coefficients, this proves propostion 4.5.1.

Exercises

1. Show that an harmonic exact form $d\gamma$ satisfies $||d\gamma||^2 = 0$. Use this to prove the uniqueness in theorem 7.2.1.
2. Prove corollary 7.2.6.
3. Fill in the details of the above example.

7.3. The Heat Equation

We will outline an approach to 7.2.5 using the heat equation due Milgram and Rosenbloom [**MR**]. However, we will deviate slightly from their presentation, which is a bit too sketchy in places. The heuristic behind the proof is that if the form α is thought of as an initial temperature, then the temperature should approach a harmonic steady state as the manifold cools. So our task is to solve the heat equation:

$$\frac{\partial A(t)}{\partial t} = -\Delta A(t) \tag{18}$$

$$A(0) = \alpha \tag{19}$$

for all $t > 0$, and study the behaviour as $t \to \infty$. For simplicial complexes, this was the content of the exercises of the first section.

We start with a few general remarks.

LEMMA 7.3.1. *If $A(t)$ is a C^∞ solution of (18), $||A(t)||^2$ is (nonstrictly) decreasing.*

PROOF. Upon differentiating $||A(t)||^2$ we obtain:

$$2\langle \frac{\partial A}{\partial t}, A\rangle = -2\langle \Delta A, A\rangle = -2(||dA||^2 + ||d^*A||^2) \leq 0.$$

□

COROLLARY 7.3.2. *A solution to (18)-(19) would be unique.*

PROOF. Given two solutions, their difference satisfies $A(0) = 0$, so it remains 0. □

The starting point for the proof of existence is the observation that when X is replaced by Euclidean space and α is a compactly supported function, then one has an explicit solution to (18)-(19)

$$A(x,t) = \int_{\mathbb{R}^n} K(x,y,t)\alpha(y)dy = \langle K(x,y,t), \alpha(y)\rangle_y$$

where the heat kernel

$$K(x,y,t) = (4\pi t)^{-n/2} e^{-||x-y||^2/4t}$$

and the symbol $\langle , \rangle_y$ is the L^2 inner product with integration carried out with respect to y.

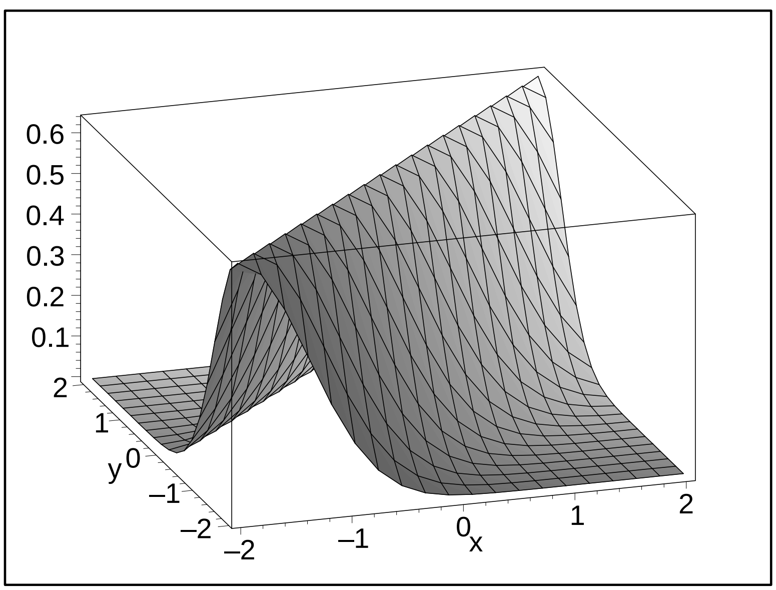


FIGURE 1. $K(x,y,0.2)$

This can be modified to make sense for a p-form on X by replacing K above by

$$K(x,y,t) = (4\pi t)^{-n/2} e^{-||x-y||^2/4t} (\sum_i dx_i \wedge dy_i)^p$$

When working with forms on $X \times X \times [0,\infty)$, we will abuse notation a bit and write them as local expressions such as $\eta(x,y,t)$. Then $d_x\eta(x,y,t)$ etcetera will indicate that the operations $d\ldots$ are preformed with y,t treated as constant (or more correctly, these operations are preformed fiberwise along a projection).

THEOREM 7.3.3. *On any compact Riemannian manifold, there exists a C^∞ p-form $K(x,y,t)$ called the heat kernel such that for any p-form α,*

$$A(x,t) = \langle K(x,y,t)\alpha(y)\rangle_y$$

gives a solution to the heat equation (18) with $A(x,0) = \alpha(x)$.

PROOF. We sketch a proof due to Minakshisundaram when $p = 0$; a more detail account can be found in [**Ch**]. This argument can be modified to work for any p [**P**, sect 4]. Let $\delta(x,y)^2$ be a nonnegative C^∞ function on $X \times X$ which agrees with the square of the Riemannian distance function in neighbourhood of the diagonal, and vanishes far away from it. Set

$$K_0(x,y,t) = (4\pi t)^{-n/2} e^{-\delta(x,y)^2/4t}$$

This is only an approximation to the true heat kernel in general, but it does satisfy a crucial property that it approaches the δ-function of "$x-y$" as $t \to 0$ in the sense that

$$\lim_{t\to 0} \langle K_0(x,y,t), \alpha(y)\rangle_y = \alpha(x)$$

However, we will need to replace K_0 by an approximation with better asymptotic properties as $t \to 0$. For each $N \in \mathbb{N}$, Minakshisundaram-Pleijel have shown how to find C^∞ functions $u_i(x,y)$ such that

$$K_1(x,y,t) = K_0(x,y,t)[u_0(x,y) + tu_1(x,y) + \ldots t^N u_N(x,y)]$$

satisfies

$$\left(\Delta_x + \frac{\partial}{\partial t}\right) K_1 = t^N K_0 \Delta_x u_N,$$

which can be written as $t^{N-n/2}e^{-\delta(x,y)^2/4t}$ times a C^∞-function. We want to emphasize that the existence of the u_i's does not involve anything deep, but rather a direct calculation. Fix $N > n/2 + 2$, and choose K_1 as above.

We define the convolution of two functions $C(x,y,t)$ and $B(x,y,t)$ by

$$(C * B)(x,y,t) = \int_0^t \langle C(x,z,t-\tau), B(z,y,\tau)\rangle_z d\tau$$

This is an associative operation. Let $L = \left(\Delta_x + \frac{\partial}{\partial t}\right) K_1$. Given a C^∞ function $B(x,y,t)$, we claim that

$$\left(\Delta_x + \frac{\partial}{\partial t}\right)(K_1 * B) = -B + L * B \tag{20}$$

Let

$$h(x,y,t,\tau) = \langle K_1(x,z,t-\tau), B(z,y,\tau)\rangle_z$$

be the integrand of $K_1 * B$. Then (20) follows formally by adding

$$\frac{\partial}{\partial t}\int_0^t h(x,y,t,\tau)d\tau = \lim_{\tau\to t} h(x,y,t,\tau) + \int_0^t \frac{\partial}{\partial t} h(x,y,t,\tau)d\tau$$

to

$$\Delta_x \int_0^t h(x,y,t,\tau)d\tau = \int_0^t \Delta_x h(x,y,t,\tau)d\tau$$

Set

$$K = K_1 + K_1 * L + K_1 * L * L + \ldots = K_1 + K_1 * (L + L * L + \ldots) \tag{21}$$

We can obtain an estimate

$$|L^{*\ell}| \le t^{N+\ell-n/2-1} C_\ell$$

where C_ℓ is a constant which goes to 0 rapidly with ℓ. This implies absolute uniform convergence of this series on $X^2 \times [0,T]$. This also shows that $K_1 * (L + L * L \ldots) \to 0$ as $t \to 0$, and therefore K also approaches the δ-function as $t \to 0$. A similar estimate on the derivatives up to second order allows us apply (20) term by term to get a telescoping series

$$\left(\Delta_x + \frac{\partial}{\partial t}\right) K = L + (-L + L * L) + (-L * L + L * L * L) + \ldots = 0$$

It follows that

$$A(x,t) = \langle K(x,y,t), \alpha(y)\rangle_y$$

satisfies the heat equation and the required initial condition. □

Let

$$T_t(\alpha) = \langle K(x,y,t), \alpha(y)\rangle_y$$

with K as in the last theorem. This is the unique solution to (18) and (19). Theorem 7.2.5 can now be deduced from:

Theorem 7.3.4 (Milgram-Rosenbloom).

1. *The semigroup property* $T_{t_1+t_2} = T_{t_1} T_{t_2}$ *holds.*
2. T_t *is formally self adjoint.*
3. $T_t \alpha$ *converges to a* C^∞ *harmonic form* $H(\alpha)$.
4. *The integral*

$$G(\alpha) = \int_0^\infty (T_t\alpha - H\alpha)dt$$

is well defined, and yields Green's operator.

Proof. We give the main ideas. The semigroup property $T_{t_1+t_2} = T_{t_1} T_{t_2}$ holds because $A(t_1 + t_2)$ can be obtained by solving the heat equation with initial condition $A(t_2)$ and then evaluating it at $t = t_1$.

To see that T_t is self adjoint, calculate

$$\frac{\partial}{\partial t}\langle T_t\eta, T_\tau\xi\rangle = \langle \frac{\partial}{\partial t} T_t\eta, T_\tau\xi\rangle = -\langle \Delta T_t\eta, T_\tau\xi\rangle$$

$$= -\langle T_t\eta, \Delta T_\tau\xi\rangle = \langle T_t\eta, \frac{\partial}{\partial \tau} T_\tau\xi\rangle = \frac{\partial}{\partial \tau}\langle T_t\eta, T_\tau\xi\rangle$$

which implies that $\langle T_t\eta, T_\tau\xi\rangle$ can be written as a function of $t + \tau$, say $g(t + \tau)$. Therefore

$$\langle T_t\eta, \xi\rangle = g(t + 0) = g(0 + t) = \langle \eta, T_t\xi\rangle.$$

The previous two properties imply that for $h \geq 0$ we have

$$\begin{aligned} ||T_{t+2h}\alpha - T_t\alpha||^2 &= ||T_{t+2h}\alpha||^2 + ||T_t\alpha||^2 - 2\langle T_{t+2h}\alpha, T_t\alpha\rangle \\ &= ||T_{t+2h}\alpha||^2 + ||T_t\alpha||^2 - 2||T_{t+h}\alpha||^2 \\ &= (||T_{t+2h}\alpha|| - ||T_t\alpha||)^2 - 2(||T_{t+h}\alpha||^2 - ||T_{t+2h}\alpha||\cdot||T_t\alpha||) \end{aligned}$$

$||T_t\alpha||^2$ converges thanks to lemma 7.3.1. Therefore $||T_{t+2h}\alpha - T_t\alpha||^2$ can be made arbitrarily small for large t. This implies that $T_t\alpha$ converges in the L^2 sense to an L^2 form $H(\alpha)$, i.e. an element of the Hilbert space completion of $\mathcal{E}^p(X)$. Fix $\tau > 0$, the relations $T_t\alpha = T_\tau T_{t-\tau}\alpha$, implies in the limit that $T_\tau H(\alpha) = H(\alpha)$. Since, T_τ is given by an integral transform with smooth kernel, it follows that $H(\alpha)$ is C^∞. The equation $T_\tau H(\alpha) = H(\alpha)$ shows that $H(\alpha)$ is in fact harmonic. We also note that H is formally self adjoint

$$\langle H\alpha, \beta\rangle = \lim_{t\to\infty}\langle T_t\alpha, \beta\rangle = \lim_{t\to\infty}\langle \alpha, T_t\beta\rangle = \langle \alpha, H\beta\rangle$$

The pointwise norms $||T_t\alpha(x) - H\alpha(x)||$ can be shown to decay rapidly enough that the integral

$$G(\alpha) = \int_0^\infty (T_t\alpha - H\alpha)dt$$

is well defined. We will verify formally that this is Green's operator:

$$\Delta G(\alpha) = \int_0^\infty \Delta T_t\alpha dt = -\int_0^\infty \frac{\partial T_t\alpha}{\partial t}dt = \alpha - H(\alpha),$$

and for β harmonic

$$\langle G(\alpha), \beta\rangle = \int_0^\infty \langle (T_t - H)\alpha, \beta\rangle dt = \int_0^\infty \langle \alpha, (T_t - H)\beta\rangle dt = 0$$

as required. □

Let's return to the example of a torus where things can be calculated explicitly.

EXAMPLE 7.3.5. *Let $X = \mathbb{R}^n/\mathbb{Z}^n$. Given α as in (17), the solution to the heat equation with initial value α is given by*

$$T_t\alpha = \sum_{\lambda\in\mathbb{Z}^n}\sum a_{\lambda,i_1\ldots i_p} e^{(2\pi i\lambda\cdot\mathbf{x} - 4\pi^2|\lambda|^2 t)} dx_{i_1}\wedge\ldots\wedge dx_{i_p},$$

and this converges to the harmonic projection

$$H(\alpha) = \sum a_{0,i_1\ldots i_p} dx_{i_1}\wedge\ldots\wedge dx_{i_p}$$

$T_t\alpha - H\alpha$ can be integrated term by term to obtain Green's operator

$$G(\alpha) = \sum_{\lambda\in\mathbb{Z}^n - \{0\}}\sum \frac{a_{\lambda,i_1\ldots i_p}}{4\pi^2|\lambda|^2} e^{2\pi i\lambda\cdot\mathbf{x}} dx_{i_1}\wedge\ldots\wedge dx_{i_p}.$$

Exercises

1. Check that the above example works as claimed.

CHAPTER 8

Toward Hodge theory for Complex Manifolds

In this chapter, we take the first few steps toward Hodge theory in the complex setting. This is really a warm up for the next chapter.

8.1. Riemann Surfaces Revisited

In this section we tie up a few loose ends from chapter 5. by proving proposition 5.4.7. Fix a compact Riemann surface X. In order to apply the techniques from the previous chapter, we need a metric which is a C^∞ family of inner products. But we need to impose a compatibility condition. As one learns in first course in complex analysis, conformal maps are angle preserving, and this means that we have a well defined notion of the angle between two tangent vectors. Among other things, compatibility will mean that the angles determined by the metric agree with the ones above. To say this more precisely, view X as two dimension real C^∞ manifold. Choosing an analytic local coordinate z in a neighbourhood of U, the vectors $v_1 = \partial/\partial x$ and $v_2 = \partial/\partial y$ give a basis (or frame) of the real tangent sheaf $\mathcal{T}_X$ of X restricted to U. The automorphism $J_p : \mathcal{T}_X|_U \to \mathcal{T}_X|_U$ represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the basis v_1, v_2 is independent of this basis, and hence globally well defined. A Riemannian metric $(,)$ is compatible with the complex structure, or is Hermitean, if the transformations J_p are orthogonal. In terms of the basis v_1, v_2 this forces the matrix of the bilinear $(,)$ form to be a positive multiple of I by some function h. Standard partition of unity arguments show that Hermitean metrics exist. Fix one. In coordinates, the metric would be represented by a tensor $h(x, y)(dx \otimes dx + dy \otimes dy)$. The volume form is represented by $hdx \wedge dy$. It follows that $*dx = dy$ and $*dy = -dx$. In other words, $*$ is the transpose of J which is independent of h. Once we have $*$, we can define all the operators from the last chapter, After having made such a fuss about metrics, it turns out that for our purposes one is as good as any other (for the statements though not the proofs).

Lemma 8.1.1. *A 1-form is harmonic if and only if its $(1,0)$ and $(0,1)$ parts are respectively holomorphic and antiholomorphic.*

Proof. A 1-form α is harmonic if and only if $d\alpha = d * \alpha = 0$. Given a local coordinate z, it can be checked that $*dz = -\sqrt{-1}dz$ and $*d\bar{z} = \sqrt{-1}d\bar{z}$. Therefore the $(1,0)$ and $(0,1)$ parts of a harmonic 1-form α are closed. If α is a $(1,0)$-form, then $d\alpha = \bar{\partial}\alpha$. Thus α is closed if and only if it is holomorphic. Conjugation yields the analogous statement for $(0,1)$-forms. □

Corollary 8.1.2. *$\dim H^0(X, \Omega^1_X)$ equals the genus of X.*

PROOF. The first Betti number $dim\, H^1(X, \mathbb{C})$ is the sum of dimensions of the spaces of holomorphic and antiholomorphic forms. Both these spaces have the same dimension, since conjugation gives a real isomorphism between them. □

LEMMA 8.1.3. *The image of* Δ *and* $\partial\bar{\partial}$ *on* $\mathcal{E}^2(X)$ *coincide.*

PROOF. Computing in local coordinates yields

$$\partial\bar{\partial} f = -\frac{\sqrt{-1}}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy$$

and

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy.$$

This finishes the proof because $\Delta = -d * d*$. □

PROPOSITION 8.1.4. *The map* $H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_X)$ *induced by* d *vanishes.*

PROOF. We use the descriptions of these spaces as $\bar{\partial}$-cohomology groups 5.4.3. Given $\alpha \in \mathcal{E}^{01}(X)$, let $\beta = d\alpha$. We have to show that β lies in the image of $\bar{\partial}$. Applying theorem 7.2.5 we can write $\beta = H(\beta) + \Delta G(\beta)$. Since β is exact, we can conclude that $H(\beta) = 0$ by corollary 7.2.6. Therefore β lies in the image of $\partial\bar{\partial} = -\bar{\partial}\partial$. □

COROLLARY 8.1.5. *The map* $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$ *is surjective, and* $\dim H^1(X, \mathcal{O}_X)$ *coincides with the genus of* X.

PROOF. The surjectivity is immediate from the long exact sequence. The second part follows from the equation

$$\dim H^1(X, \mathcal{O}_X) = \dim H^1(X, \mathbb{C}) - \dim H^0(X, \Omega_X)$$

□

Exercises

1. Show that $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$ can be identified the projection of harmonic 1-forms to antiholomorphic $(0, 1)$-forms.
2. Calculate the spaces of harmonic and holomorphic one forms explicitly for an elliptic curve.
3. Show that the pairing $(\alpha, \beta) \mapsto \int_X \alpha \wedge \overline{\beta}$ is positive definite.

8.2. Dolbeault's theorem

We now extend the results from Riemann surfaces to higher dimensions. Given an n dimensional complex manifold X, let $\mathcal{O}_X$ denote the sheaf of holomorphic functions. We can regard X as a $2n$ dimensional (real) C^∞ manifold as explained in section 1.2. As with Riemann surfaces, from now on C^∞_X will denote the sheaf of complex valued C^∞ functions. Likewise $\mathcal{E}^k_X$ will denote the sheaf of C^∞ complex valued k-forms, but first we should make clear what that means. Let $\mathcal{E}^k_{X,\mathbb{R}}$ to denote the sheaf of real valued C^∞ k-forms as defined in section 1.5. Then

$$\mathcal{E}^k_X(U) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{E}^k_{X,\mathbb{R}}(U).$$

By a *real structure* on a complex vector space V, we mean real vector space $V_{\mathbb{R}}$ and an isomorphism $\mathbb{C}\otimes V_{\mathbb{R}} \cong V$. This gives rise to a $\mathbb{C}$-antilinear involution $v \mapsto \bar{v}$ given by $\overline{a \otimes v} = \bar{a} \otimes v$. Conversely, such an involution gives rise to the real structure $V_{\mathbb{R}} = \{v \mid \bar{v} = v\}$. In particular, $\mathcal{E}^k_X(U)$ has a natural real structure.

The sheaf of holomorphic p-forms Ω^p_X is a subsheaf of $\mathcal{E}^p_X$ stable under multiplication by $\mathcal{O}_X$. This sheaf is locally free as an $\mathcal{O}_X$-module. If $z_1, \ldots z_n$ are holomorphic coordinates defined on an open set $U \subset X$,

$$\{dz_{i_1} \wedge \ldots \wedge dz_{i_p} \mid i_1 < \ldots < i_p\}$$

gives a basis for $\Omega^p_X(U)$. To simplify our formulas, we will write the above expressions as dz_I where $I = \{i_1, \ldots, i_p\}$.

Definition 8.2.1. *Let $\mathcal{E}^{(p,0)}_X$ denote the C^∞ submodule of $\mathcal{E}^p$ generated by Ω^p_X. Let $\mathcal{E}^{(0,p)}_X = \overline{\mathcal{E}^{(p,0)}_X}$, and $\mathcal{E}^{(p,q)}_X = \mathcal{E}^{(p,0)}_X \wedge \mathcal{E}^{(0,q)}_X$.*

In local coordinates $\{dz_I \wedge d\bar{z}_J \mid \#I = p, \#J = q\}$, gives a basis of $\mathcal{E}^{(p,q)}_X(U)$. All of the operations of sections 5.3 and 5.4 can be extended to the higher dimensional case. The operators

$$\partial : \mathcal{E}^{(p,q)}_X \to \mathcal{E}^{(p+1,q)}_X$$

and

$$\bar{\partial} : \mathcal{E}^{(p,q)}_X \to \mathcal{E}^{(p,q+1)}_X$$

are given locally by

$$\partial(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J) = \sum_{I,J} \sum_{i=1}^{n} \frac{\partial f_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J$$

$$\bar{\partial}(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J) = \sum_{I,J} \sum_{j=1}^{n} \frac{\partial f_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J$$

The identities

$$\begin{aligned} d &= \partial + \bar{\partial} \\ \partial^2 &= \bar{\partial}^2 = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial &= 0 \end{aligned} \tag{22}$$

hold.

The analogue of theorem 5.3.2 is

Theorem 8.2.2. *Let $D \subset \mathbb{C}^n$ be an open polydisk (i.e. a product of disks). Given $\alpha \in \mathcal{E}^{(p,q)}(\bar{D})$ with $\bar{\partial}\alpha = 0$, there exists $\beta \in C^\infty(D)$ such that $\alpha = \bar{\partial}\beta$*

Proof. See [**GH**, pp. 25-26]. □

Corollary 8.2.3 (Dolbeault's Theorem).

$$0 \to \Omega^p_X \to \mathcal{E}^{(p,0)}_X \xrightarrow{\bar{\partial}} \mathcal{E}^{(p,1)}_X \ldots$$

is a fine resolution.

PROOF. Given a $(p,0)$ form $\sum_I f_I dz_I$,

$$\bar{\partial}(\sum_I f_I dz_I) = \sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I = 0$$

if and only it is holomorphic. In other words, Ω_X^p is the kernel of the $\bar{\partial}$ operator on $\mathcal{E}_X^{(p,0)}$, and the corollary follows.

□

Exercises

1. Check the identities 22 when $\dim X = 2$.

8.3. Complex Tori

A complex torus is a quotient $X = V/L$ of a finite dimensional complex vector space by a lattice (i.e. a discrete subgroup of maximal rank). Thus it is both a complex manifold and a torus. Let us identify V with $\mathbb{C}^n$. Let $z_1, \dots z_n$ be the standard complex coordinates on $\mathbb{C}^n$, and let $x_i = Re(z_i)$, $y_i = Im(z_i)$.

We have already seen that harmonic forms with respect to the flat, i.e. Euclidean, metric are the forms with constant coefficients. The space of harmonic forms can be decomposed into (p,q) type.

$$H^{(p,q)} = \bigoplus_{\#I=p, \#J=q} \mathbb{C} dz_I \wedge d\bar{z}_J$$

These forms are certainly $\bar{\partial}$-closed, and hence define $\bar{\partial}$-cohomology classes.

PROPOSITION 8.3.1. $H^{(p,q)} \cong H^q(X, \Omega_X^p)$.

The isomorphism $V \cong L \otimes \mathbb{R}$ induces a natural real structure on V. Therefore it makes sense to speak of antilinear maps from V to $\mathbb{C}$, these are maps $f(av) = \bar{a} f(v)$; let $\bar{V}^*$ denote the space of these.

COROLLARY 8.3.2. $H^q(X, \Omega_X^p) \cong \wedge^q V \otimes \wedge^p \bar{V}^*$

Let ∂^* and $\bar{\partial}^*$ denote the adjoints to ∂ and $\bar{\partial}$ respectively. These operators can be calculated explicitly. Let i_k and $\bar{i}_k$ denote contraction with the vector fields $2\partial/\partial z_k$ and $2\partial/\partial \bar{z}_k$. These are the adjoints to $dz_k \wedge$ and $d\bar{z}_k \wedge$. Then

$$\partial^* \alpha = -\sum \frac{\partial}{\partial z_k} i_k \alpha$$

$$\bar{\partial}^* \alpha = -\sum \frac{\bar{\partial}}{\bar{\partial} z_k} i_k \alpha$$

We can define the ∂ and $\bar{\partial}$-Laplacians by

$$\Delta_\partial = \partial^* \partial + \partial \partial^*,$$

$$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

LEMMA 8.3.3. $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$.

This can be checked directly by an explicit calculation of the Laplacians [**GH**, p. 106]. However, we will somewhat perversely sketch a more complicated proof, since it this argument which generalizes.

Introduce the auxillary operators

$$\omega = \frac{\sqrt{-1}}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k$$
$$L\alpha = \omega \wedge \alpha$$

and

$$\Lambda = -\frac{\sqrt{-1}}{2}\sum \bar{i}_k i_k$$

A straight forward computation yields the first order Kähler identities:

PROPOSITION 8.3.4. *If* $[A, B] = AB - BA$ *then*

1. $[\Lambda, \bar{\partial}] = -i\partial^*$
2. $[\Lambda, \partial] = i\bar{\partial}^*$

PROOF. The proof is a long but elementary calculation, [**GH**, p. 114]. □

Upon substituting these into the definitions of the various Laplacians some remarkable cancelations take place, and we obtain:

PROOF OF LEMMA 8.3.3. We first establish $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$,

$$i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) =$$
$$= \partial(\Lambda\partial - \partial\Lambda) - (\Lambda\partial - \partial\Lambda)\partial$$
$$= \partial\Lambda\partial - \partial\Lambda\partial = 0$$

Similarly, $\partial^*\bar{\partial} + \bar{\partial}\partial^* = 0$.

Next expand Δ,

$$\Delta = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})$$
$$= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\partial^*\bar{\partial} + \bar{\partial}\partial^*)$$
$$= \Delta_\partial + \Delta_{\bar{\partial}}$$

Finally, we check $\Delta_\partial = \Delta_{\bar{\partial}}$,

$$-i\Delta_\partial = \partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) + (\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial$$
$$= \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial$$
$$= (\partial\Lambda - \Lambda\partial)\bar{\partial} + \bar{\partial}(\partial\Lambda - \Lambda\partial) = -i\Delta_{\bar{\partial}}$$

□

PROOF OF PROPOSITION 8.3.1. By Dolbeault's theorem

$$H^q(X, \Omega_X^p) \cong \frac{ker[\bar{\partial} : \mathcal{E}^{(p,q)}(X) \to \mathcal{E}^{(p,q+1)}]}{im[\bar{\partial} : \mathcal{E}^{(p,q-1)}(X) \to \mathcal{E}^{(p,q)}]}$$

Let α be a $\bar{\partial}$-closed (p, q)-form. Decompose

$$\alpha = \beta + \Delta\gamma = \beta + 2\Delta_{\bar{\partial}}\gamma = \beta + \bar{\partial}\gamma_1 + \bar{\partial}^*\gamma_2$$

with β harmonic, which possible by theorem7.2.5. We have

$$||\bar{\partial}^*\gamma_2||^2 = \langle\gamma_2, \bar{\partial}\bar{\partial}^*\gamma_2\rangle = \langle\gamma_2, \bar{\partial}\alpha\rangle = 0.$$

It is left as an exercise to check that β is of type (p,q), and that it is unique Therefore the $\bar{\partial}$-class of α has a unique representive by a constant (p,q)-form. $\square$

Exercises

1. Show that that $\beta + \bar{\partial}\gamma = 0$ forces $\beta = 0$ if β is harmonic.
2. Suppose that $\alpha = \beta + \bar{\partial}\gamma$ and that α is of type (p,q) and β harmonic. By decomposing $\beta = \sum \beta^{(p',q')}$ and $\gamma = \sum \gamma^{(p',q')}$ into (p',q') type and using the previous exercise, prove that β is of type (p,q) and unique.

CHAPTER 9

Kähler manifolds

In this chapter, we extend the results of the previous section to an important class of manifolds called Kähler manifolds.

9.1. Kähler metrics

Let X be a compact complex manifold with complex dimension n. Fix a Hermitean metric H, which is a choice of Hermitean inner product on the complex tangent spaces which vary in C^∞ fashion. More precisely, H would be given by a section of the $\mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)}$, sthat in some (any) locally coordinate system around each point is given by

$$H = \sum h_{ij} dz_i \otimes d\bar{z}_j$$

with h_{ij} positive definite Hermitean. By linear algebra, the real and imaginary parts of h_{ij} are respectively symmetric positive definite and nondegenerate skew symmetric matrices. Geometrically, the real part is just a Riemannian structure, while the (suitably normalized) imaginary part of gives a $(1,1)$-form ω called the *Kähler form*; in local coodinates

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j.$$

It is clear from this formula, that ω determines the metric.

Definition 9.1.1. *A Hermitean metric on X is called Kähler if there exist analytic coordinates $z_1, \dots z_n$ about any point, for which the metric becomes Euclidean up to second order:*

$$h_{ij} \equiv \delta_{ij} \mod (z_1, \dots z_n)^2$$

A Kähler manifold is a complex manifold which admits a Kähler metric.

In such a coordinate system,

$$\omega = \frac{\sqrt{-1}}{2} \sum dz_i \wedge d\bar{z}_i + \text{ terms of 2nd order and higher}$$

Therefore $d\omega = 0$. Thus not every Hermitean metric is Kähler. This condition characterizes Kähler metrics (and usually taken as the definition):

Proposition 9.1.2. *Given a Hermitean metric h, the following are equivalent*

1. *h is Kähler.*
2. *The Kähler form is closed: $d\omega = 0$.*
3. *The Kähler form is locally expressable as $\omega = \partial\bar{\partial} f$.*

Proof. See [**GH**, p 107] for the equivalence of the first two conditions, and the exercises for the last two. □

We will refer the cohomology class of ω as the *Käher class*. The function f in the proposition is called a Kähler potential. A function f is *plurisubharmonic* if it is Kähler potential, or equivalently in coordinates this means

$$\sum \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j > 0$$

for any nonzero vector ξ.

Standard examples of compact Kähler manifolds are:

Example 9.1.3. *Riemann surfaces with any Hermitean metric since $d\omega$ vanishes for trivial reasons.*

Example 9.1.4. *Complex tori with flat metrics.*

Example 9.1.5. *$\mathbb{P}^n$ with the Fubini-Study metric. This is the unique metric with Kähler form which pulls back to*

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(|z_0|^2 + \ldots |z_n|^2)$$

under the canonical map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$. Since $H^2(\mathbb{P}^n, \mathbb{C})$ is one dimensional, the Kähler class $[\omega]$ would have to be a nonzero multiple of $c_1(\mathcal{O}(1))$. In fact, the constants in the formulas are chosen *so that these coincide.*

Definition 9.1.6. *A line bundle L on a compact complex manifold X is called if called very ample if there is an embedding $X \subset \mathbb{P}^n$ such that $L \cong \mathcal{O}_{\mathbb{P}^n}(1)|_X$. H is ample if some positive tensor power of it is very ample.*

The key examples of Kähler manifolds are provided by

Lemma 9.1.7. *A complex submanifold of a Kähler manifold inherits a Kähler metric such that the Kähler class is the restriction of the Kähler class of the ambient manifold.*

Proof. This comes down to the fact that a plurisubharmonic function will restrict to plurisubharmonic function on complex submanifold. Thus a Kähler form will restrict to a Kähler form. □

Corollary 9.1.8. *A smooth projective variety has a Kähler metric, with Kähler class equal to $c_1(L)$ for L very ample.*

Since $c_1(L^{\otimes n}) = nc_1(L)$, we can replace "very ample" by "ample" above.

Exercises

1. Prove the equivalence of 2 and 3 in proposition 9.1.2. For 2⇒ 3, use Poincaré's lemma to locally solve $\omega = d\alpha$ and the $\bar{\partial}$-Poincaré lemma (8.2.2) to the $(0,1)$-part of α and the conjugate of $(1,0)$-part.
2. Check that the Fubini-Study metric is Kähler.

9.2. The Hodge decomposition

Fix a Kähler manifold X with a specific metric. Since Kähler metrics are Euclidean up to second order, we have the following metatheorem: *Any identity involving geometrically defined first order differential operators on Euclidean space* *will automatically extend to Kähler manifolds.* Let us introduce the relevant operators. X has a canonical orientation, so the Hodge star operator associated to the Riemannian structure can be defined. $*$ will be extended to $\mathbb{C}$-linear operator on $\mathcal{E}^\bullet_X$, and set $\bar{*}(\alpha) = \overline{*\alpha} = *\bar{\alpha}$[1]. $*$ is compatible with the natural bigrading on forms in the sense that

$$*\mathcal{E}^{(p,q)}(X) \subseteq \mathcal{E}^{(n-q,n-p)}(X).$$

Let $\partial^* = -\bar{*}\partial\bar{*}$ and $\bar{\partial}^* = -\bar{*}\bar{\partial}\bar{*}$. These are adjoints of ∂ and $\bar{\partial}$. Then we can define the operators

$$\Delta_\partial = \partial^*\partial + \partial\partial^*,$$
$$\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$
$$L = \omega\wedge$$
$$\Lambda = - * L *$$

A form is called $\bar{\partial}$-harmonic if it lies in the kernel of $\Delta_{\bar{\partial}}$. For a general Hermitean manifold there is no relation between harmonicity and $\bar{\partial}$-harmonicity. However, these notions do coincide for the class the class of Kähler manifolds, which we will define shortly.

Lemma 8.3.3 generalizes:

THEOREM 9.2.1. $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$.

PROOF. Since Laplacians are second order, the above metatheorem will not apply directly. However, proposition 8.3.4 extends to X by the above principle. The rest of the argument is the same as in section 8.3. □

COROLLARY 9.2.2. *If X is compact then $H^q(X, \Omega^p_X)$ is isomorphic the space of harmonic (p,q)-forms.*

PROOF. Since harmonic forms are $\bar{\partial}$-closed (exercise), we have map from the space of harmonic (p,q)-forms to $H^q(X, \Omega^p_X)$. The rest of the argument is identical to thhe proof of proposition 8.3.1. □

We obtain the following special case of Serre duality as a consequence:

COROLLARY 9.2.3. *When X is compact, $H^p(X, \Omega^q_X) \cong H^{n-p}(X, \Omega^{n-q}_X)$*

PROOF. $\bar{*}$ induces an $\mathbb{R}$-linear isomorphism between the corresponding spaces of harmonic forms. □

THEOREM 9.2.4 (The Hodge decomposition). *If X is a compact Kähler manifold then a differential form is harmonic if and only if its (p,q) components are. Consequently we have (for the moment) noncanonical isomorphisms*

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega^p_X).$$

Furthermore, complex conjugation induces $\mathbb{R}$-linear isomorphisms between the space of harmonic (p,q) and (q,p) forms. Therefore

$$H^q(X, \Omega^p_X) \cong H^p(X, \Omega^q_X).$$

[1] Many authors, notably Griffiths and Harris [GH], interchange these operators.

PROOF. The operator $\Delta_{\bar\partial}$ preserves the decomposition

$$\mathcal{E}^i(X) = \bigoplus_{p+q=i} \mathcal{E}^{(p,q)}(X)$$

Therefore a form is harmonic if and only if its (p,q) components are. Since complex conjugation commutes with Δ, conjugation preserves harmonicity. These arguments, together with corollary 9.2.2, finishes the proof. □

COROLLARY 9.2.5. *The Hodge numbers $h^{pq}(X) = dim H^q(X, \Omega^p_X)$ are finite dimensional.*

COROLLARY 9.2.6. *If i is odd then the ith Betti number b_i of X is even.*

PROOF.

$$b_i = 2\sum_{p<q} h^{pq}$$

□

The first corollary is true for compact complex non Kähler manifolds, however the second may fail (exerice). The Hodge numbers gives an important set of holomorphic invariants for X. We can visualize them by arranging them in a diamond:

$$\begin{array}{ccccccc} & & & h^{00}=1 & & & \\ & & h^{10} & & h^{01} & & \\ h^{20} & & & h^{11} & & & h^{02} \\ & & & \ldots & & & \\ & & h^{n,n-1} & & h^{n-1,n} & & \\ & & & h^{nn}=1 & & & \end{array}$$

The previous results implies that this picture has both vertical and lateral symmetry (e.g $h^{10} = h^{01} = h^{n,n-1} = h^{n-1,n}$).

Exercises

1. Prove that α is harmonic if and only if $\bar\partial\alpha = \bar\partial^*\alpha = 0$.
2. The Hopf surface S is complex manifold obtained as a quotient of $\mathbb{C}^2 - \{0\}$ by $\mathbb{Z}$ acting by $z \mapsto 2^n z$. Show that S is homeomorphic to $S^1 \times S^3$, and conclude that it cannot be Kähler.
3. Using the metatheorem, verify that $dvol = \frac{\omega^2}{2}$ on a 2 dimensional Kähler manifold. Conclude that $[\omega]$ is nonzero.
4. Generalize the previous exercise to all dimensions.

9.3. Picard groups

Let us write $H^2(X,\mathbb{Z}) \cap H^{11}(X)$ for the preimage of $H^{11}(X)$ under the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$. Note that this contains the torsion subgroup $H^2(X,\mathbb{Z})_{tors}$.

THEOREM 9.3.1. *Let X be compact Kähler manifold. Then $Pic(X)$ fits into an exact sequence*

$$0 \to Pic^0(X) \to Pic(X) \to H^2(X,\mathbb{Z}) \cap H^{11}(X) \to 0$$

with $Pic^0(X)$ a complex torus of dimension $h^{01}(X)$

There are two assertions which will be proved seperately.

PROPOSITION 9.3.2. *$Pic^0(X) = ker(c_1)$ is torus.*

PROOF. From the exponential sequence, we obtain

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to Pic(X) \to H^2(X,\mathbb{Z})$$

Pic^0 is the cokernel of the first map. This map can be factored through $H^1(X,\mathbb{R}_X)$, and certainly $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$ is a torus. Thus for the first part, it suffices to prove that

$$\pi : H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X)$$

is an isomorphism of real vector spaces. We know that that these space have the same real dimension $b_1 = 2h^{10}$, so it is enough to check that π is injective. The Hodge decomposition implies that $\alpha \in H^1(X,\mathbb{R})$ can be represented by a sum of a harmonic $(1,0)$-form α_1 and a harmonic $(0,1)$-form α_2. Since $\alpha = \bar{\alpha}$, $\alpha_1 = \bar{\alpha}_2$. $\pi(\alpha)$ is just α_1. Therefore $\pi(\alpha) = 0$ implies that $\alpha = 0$. □

PROPOSITION 9.3.3 (The Lefschetz $(1,1)$ theorem). *$c_1(Pic(X)) = H^2(X,\mathbb{Z}) \cap H^{11}(X)$*

PROOF. The map $f : H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ can be factored as $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ followed by projection of the space of harmonic 2-forms to the space of harmonic $(2,0)$-forms. From the exponential sequence $c_1(Pic(X)) = ker(f)$, and this certainly contains $H^2(X,\mathbb{Z}) \cap H^{11}(X)$.

Conversely suppose that $\alpha \in ker(f)$, then its image in $H^2(X,\mathbb{C})$ can be represented by a sum of a harmonic $(0,2)$-form α_1 and a harmonic $(1,1)$-form α_2. Since $\bar{\alpha} = \alpha$, α_1 must be zero. Therefore $\alpha \in H^2(X,\mathbb{Z}) \cap H^{11}(X)$. □

$Pic^0(X)$ is called the Picard torus (or variety when X is projective). When X is a compact Riemann surface, $Pic^0(X)$ is usually called the Jacobian and denoted by $J(X)$

EXAMPLE 9.3.4. *When $X = V/L$ is a complex torus, $Pic^0(X)$ is a new torus called the dual torus. Using corollary 8.3.2, it can be seen to be isomorphic to $\bar{V}^*/L^*$, where $\bar{V}^*$ is the antilinear dual and*

$$L^* = \{\lambda \in V^* \,|, Im(\lambda)(L) \subseteq \mathbb{Z}\}$$

The dual of the Picard torus is called the Albanese torus $Alb(X)$. Since $H^{01} = \bar{H}^{01}$, $Alb(X)$ is isomorphic to

$$\frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})/H_1(X,\mathbb{Z})_{tors}},$$

where the elements γ of the denominator are identified with the functionals $\alpha \mapsto \int_\gamma \alpha$ in numerator.

Exercises

1. Let X be a compact Riemann surface. Show that the pairing $<\alpha,\beta> = \int_X \alpha\wedge\beta$ induces an isomorphism $J(X) \cong Alb(X)$; in other words $J(X)$ is self dual.
2. Fix a base point $x_0 \in X$. Show that the so called Abel-Jacobi map $X \to Alb(X)$ given by $x \mapsto \int_{x_0}^x$ (which is well defined modulo H_1) is holomorphic.

3. Show that every holomorphic 1-form on X is the pullback of a holomorphic 1-form from $Alb(X)$. In particular, the Abel-Jacobi map cannot be constant if $h^{01} \neq 0$.

CHAPTER 10

Homological methods in Hodge theory

We introduce some homological tools which will allow us to extend and refine the results of the previous chapter.

10.1. Pure Hodge structures

It is useful isolate the purely linear algebraic features of the Hodge decomposition. We define a *pure real Hodge structure* of weight m to be finite dimensional complex vector space with a real structure $H_{\mathbb{R}}$, and hence a conjugation, and a bigrading

$$H = \bigoplus_{p+q=m} H^{pq}$$

satisfying $\bar{H}^{pq} = H^{qp}$. We generally use the same symbol for Hodge structure and the underlying vector space. A (pure weight m) Hodge structure is a real Hodge structure H together with a choice of a finitely generated abelian group $H_{\mathbb{Z}}$ and an isomorphism $H_{\mathbb{Z}} \otimes \mathbb{R} \cong H_{\mathbb{R}}$. Even though the abelian group $H_{\mathbb{Z}}$ may have torsion, it is helpful to think of it as "lattice" in $H_{\mathbb{R}}$. Rational Hodge structures are defined in a similar way. Given a pure Hodge structure, define the Hodge filtration by

$$F^p H_{\mathbb{C}} = \bigoplus_{p' \geq p} H^{pq}.$$

The following is elementary.

Lemma 10.1.1. *If H is a pure Hodge structure of weight m then*

$$H_{\mathbb{C}} = F^p \bigoplus \bar{F}^{m-p+1}$$

for all p. Conversely if $F^\bullet$ is a descending filtration satisfying $F^a = H_{\mathbb{C}}$ and $F^b = 0$ for some $a, b \in \mathbb{Z}$ and satisfying the above identity, then

$$H^{pq} = F^p \cap \bar{F}^q$$

defines a pure Hodge structure of weight m.

The most natural examples come from the mth cohomology of a compact Kähler manifold, where we take $H_{\mathbb{Z}}$ is the image of $H^m(X, \mathbb{Z})$. But it is easy to manufacture other examples. For every integer i, there is a rank one Hodge structure $\mathbb{Z}(i)$ of weight $-2i$ Here the underlying space is $\mathbb{C}$, with $H^{(-i,-i)} = \mathbb{C}$, and lattice $H_{\mathbb{Z}} = (2\pi\sqrt{-1})^i\mathbb{Z}$ (these normalizations should be ignored at first). The collection of Hodge structures of forms a category HS, where a morphism is linear map f preserving the lattices and the bigradings. In particular, morphisms between Hodge structure with different weights must vanish. This category has the following operations: direct sums with the restriction that we can add Hodge structures of the same weight (we will eventually relax this), and (unrestricted) tensor products and duals. Explicitly, given Hodge structures H and G of weights n and m. their tensor product $H \otimes_{\mathbb{Z}} G$ is equipped with a weight $n+m$ Hodge structure with bigrading

$$(H \otimes G)^{pq} = \bigoplus_{p'+p''=p, q'+q''=q} H^{p'q'} \otimes G^{p''q''}.$$

If $m = n$, their direct sum $H \oplus G$ is equipped with the weight m Hodge structure

$$(H \oplus G)^{pq} = \bigoplus_{p+q=m} H^{pq} \oplus G^{pq}.$$

The dual $H^* = Hom(H, \mathbb{Z})$ is equipped with a weight $-n$ Hodge structure with bigrading

$$(H^*)^{pq} = (H^{-p,-q})^*.$$

The operation $H \mapsto H(i)$ is called the Tate twist. It has the effect of leaving H unchanged and shifting the bigrading by $(-i, -i)$.

The obvious isomorphism invariants of a Hodge structure H are its Hodge numbers $\dim H^{pq}$. However, this the doesn't completely characterize them. Consider, the set of $2g$ dimensional Hodge structures H of weight 1 and level 1. This means that the Hodge numbers are as follows: $\dim H^{10} = \dim H^{01} = g$ and the others zero. There are uncountably many isomorphism classes, in fact:

LEMMA 10.1.2. *There is a one to one correspondence between isomorphism classes of Hodge structures as above and g dimensional complex tori given by*

$$H \mapsto \frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^1}$$

Exercises

1. If H is a weight one (not necessarily) level one Hodge structure, show that the construction of lemma 10.1.2 is a torus.
2. Prove lemma 10.1.2.

10.2. Canonical Hodge Decomposition

The Hodge decomposition involved harmonic forms so it is tied up with the Käher metric. It is possble to reformulate it, so as to make it independent of the choice of metric. Let us see how this works for a compact Riemann surface X. We have an exact sequence

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

and we saw in lemma 5.4.6 that the induced map

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

is injective. If we define

$$\begin{aligned} F^0 H^1(X, \mathbb{C}) &= H^1(X, \mathbb{C}) \\ F^1 H^1(X, \mathbb{C}) &= im(H^0(X, \Omega^1_X)) \\ F^2 H^1(X, \mathbb{C}) &= 0 \end{aligned}$$

then this together with the isomorphism $H^1(X, \mathbb{C}) = H^1(X, \mathbb{Z}) \otimes \mathbb{C}$ determines a pure Hodge structure of weight 1. To see this choose a metric which is automatically Kähler because $dimX = 1$ (in fact $*$ is a conformal invariant so it's not necessary to choose a metric at all). Then $H^1(X,\mathbb{C})$ is isomorphic to direct sum of the space of harmonic $(1,0)$-forms which maps to F^1 and the space of harmonic $(0,1)$ forms which maps to $\bar{F}^1$.

Before proceeding with the higher dimensional version, we need some facts from homological algebra. Let

$$C^\bullet = \ldots C^a \xrightarrow{d} C^{a+1} \to \ldots$$

be a complex of vectors spaces (or modules or...). It is convenient to all the indices to vary over $\mathbb{Z}$, but we will require that it is *bounded below* which means that $C^a = 0$ for all $a << 0$. Let us suppose that each C^i is equipped with a filtration $F^pC^i \supseteq F^{p+1}C^i \ldots$ which is preserved by d, i.e. $dF^pC^i \subseteq F^pC^{i+1}$. This implies that each $F^pC^\bullet$ is a subcomplex. We need to impose a finiteness condition, $F^\bullet$ *biregular* if for each i there exists a and b with $F^aC^i = C^i$ and $F^bC^i = 0$. We get a map on cohomology

$$\phi^p : H^\bullet(F^pC^\bullet) \to H^\bullet(C^\bullet)$$

and we let $F^pH^\bullet(C^\bullet)$ be the image. The filtration is said to be *strictly compatible with differentials* of $C^\bullet$, or simply just *strict* if all the ϕ's are injective. Let $Gr^p_F C^\bullet = Gr^pC^\bullet = F^pC^\bullet/F^{p+1}C^\bullet$. Then we have a short exact sequence of complexes:

$$0 \to F^{p+1}C \to F^pC \to Gr^pC \to 0$$

From which we get connecting maps $\delta : H^i(Gr^pC^\bullet) \to H^{i+1}(F^{p+1}C^\bullet)$. This can be described explicitly as follows. Given $\bar{x} \in H^i(Gr^pC^\bullet)$, it can be lifted to an element $x \in F^pC^i$ such that $dx \in F^{p+1}C^{i+1}$. Then $\delta(\bar{x})$ is represented by dx.

PROPOSITION 10.2.1. *The following are equivalent*

1. *F is strict.*
2. *$F^pC^{i+1} \cap dC^i = dF^pC^i$ for all i and p.*
3. *The connecting maps $\delta : H^i(Gr^pC^\bullet) \to H^{i+1}(F^{p+1}C^\bullet)$ vanish for all i and p.*

PROOF. We first show (1) $\Rightarrow$ (2). Suppose that $z \in F^pC^{i+1} \cap dC^i$, then $z = dx$ with $x \in C^i$ lies in F^p. Clearly dx defines an element of $ker\phi^p$. Therefore $dx = dy$ for some $y \in F^p$ since F is strict.

Now assume (2). Given $\bar{x} \in H^i(Gr^p)$, lift it to an element $x \in F^p$ with $dx \in F^{p+1}$ as above. Then $dx = dy$ for some $y \in F^{p+1}$. Replacing x by $x - y$ shows that $\delta(\bar{x})$ is zero.

(3) is equivalent to the injectivity of all the maps $H^i(F^{p+1}) \to H^i(F^p)$, and this implies (1). □

Define $Gr^pH^i(C^\bullet) = F^pH^i(C^\bullet)/F^{p+1}H^i(C^\bullet)$. For vector spaces, there are noncanonical isomorphisms

$$H^i(C^\bullet) = \bigoplus_p Gr^pH^i(\mathbb{C}^\bullet)$$

COROLLARY 10.2.2. *$Gr^pH^i(C^\bullet)$ is a subquotient of$H^i(Gr^pC^\bullet)$. This means that there is a diagram*

$$H^i(Gr^pC^\bullet) \supseteq I^{i,p} \to Gr^pH^i(C^\bullet)$$

where the last map is onto. Isomorphisms $Gr^pH^i(C^\bullet) \cong I^{i,p} \cong H^i(Gr^pC^\bullet)$ hold, for all i, p, if and only if F is strict.

PROOF. Let $I^{i,p} = im[H^i(F^p) \to H^i(Gr^p C^\bullet)]$ then the surjection $H^i(F^p) \to Gr^p H^i(C^\bullet)$ factors through I. The remaining statement follows from (3) and a diagram chase. □

COROLLARY 10.2.3. *Suppose that $C^\bullet$ is a complex of vector spaces over a field such that $dim H^i(Gr^p) < \infty$ for all i, p. Then*

$$\dim H^i(C^\bullet) \leq \sum_p \dim H^i(Gr^p),$$

and equality holds for all i if and only if F is strict. In which case, we also have

$$\dim F^p H^i(C^\bullet) = \sum_{p' \geq p} \dim H^i(Gr^{p'}).$$

PROOF. We have

$$\dim H^i(C^\bullet) = \sum_p \dim Gr^p H^i(C^\bullet) \leq \sum_p \dim I^{ip} \leq \sum_p \dim H^i(Gr^p)$$

and equallity is equivalent to strictness of F by the previous corollary. The last statement is left as an exercise. □

These results are usually formulated in terms of spectral sequences which we have chosen to avoid. In this language the last corollary says that F is strict if and only if the associated spectral sequence degenerates at E_1. The notation is partially explained in the exercises.

Let X be a complex manifold, then the de Rham complex $\mathcal{E}^\bullet(X)$ has a filtration called the Hodge filtration:

$$F^p \mathcal{E}^\bullet(X) = \sum_{p' \geq p} \mathcal{E}^{p'q}(X)$$

Its conjugate equals

$$\bar{F}^q \mathcal{E}^\bullet(X) = \sum_{q' \geq q} \mathcal{E}^{pq'}(X)$$

THEOREM 10.2.4. *If X is compact Kähler, the Hodge filtration is strict. The associated filtration $F^\bullet H^i(X, \mathbb{C})$ gives a canonical Hodge structure*

$$H^i(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{pq}(X)$$

of weight i, where

$$H^{pq}(X) = F^p H^i(X, \mathbb{C}) \cap \bar{F}^q H^i(X, \mathbb{C}) \cong H^q(X, \Omega^p_X).$$

PROOF. $H^q(Gr^p \mathcal{E}^\bullet(X)) = H^q(\mathcal{E}^{(p,\bullet)}_X(X))$ is isomorphic to $H^q(X, \Omega^p_X)$. Therefore F is strict by corollary 10.2.3 and the Hodge decomposition. By conjugation, we see that $\bar{F}$ is also strict. Thus

$$\dim F^p H^i(X, \mathbb{C}) = h^{p,i-p}(X) + h^{p+1,i-p-1}(X) + \dots$$

and

$$\dim \bar{F}^{i-p+1} H^i(X, \mathbb{C}) = h^{p-1,i-p+1}(X) + h^{p-2,i-p+2}(X) + \dots$$

These dimensions add up to the ith Betti number. A cohomology class lies in $F^p H^i(X, \mathbb{C})$ (respectively $\bar{F}^{i-p+1} H^i(X, \mathbb{C})$) if and only if it can be represented by a form in $F^p \mathcal{E}^\bullet(X)$ (respectively $\bar{F}^{i-p+1} \mathcal{E}^\bullet(X)$). Thus $H^i(X, \mathbb{C})$ is the sum of these subspaces, and it must necessarily be a direct sum. Therefore the filtrations determine a pure Hodge structure of weight i on $H^i(X, \mathbb{C})$. □

Even though harmonic theory lies is behind this. It should be clear that the final result does not involve the metric. The following is not so much a corollary as an explanation of what the term canonical means:

COROLLARY 10.2.5. *If $f : X \to Y$ is a holomorphic map of compact Kähler manifolds, then the pullback map $f^* : H^i(Y, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is compatible with the Hodge structures.*

COROLLARY 10.2.6. *Global holomorphic differential forms on X are closed.*

PROOF. Strictness implies that the maps

$$d : H^0(X, \Omega^p_X) \to H^0(X, \Omega^{p+1}_X)$$

vanish by the exercises. □

This corollary, and hence the theorem, can fail for compact complex non Kähler manifolds [**GH**, p. 444].

THEOREM 10.2.7. *If X is a compact Kähler manifold, the cup product*

$$H^i(X) \otimes H^j(X) \to H^{i+j}(X)$$

is a morphism of Hodge structures.

The proof comes down to the observation that

$$F^p\mathcal{E}^\bullet \wedge F^q\mathcal{E}^\bullet \subseteq F^{p+q}\mathcal{E}^\bullet$$

For the corollaries, we work with rational Hodge structures. We have compatability with Poincaré duality:

COROLLARY 10.2.8. *If $\dim X = n$, then Poincaré duality gives an isomorphism of Hodge structures*

$$H^i(X) \cong [H^{2n-i}(X)^*](-n)$$

We have compatibility with the Künneth formula:

COROLLARY 10.2.9. *If X and Y are compact manifolds, then*

$$\bigoplus_{i+j=k} H^i(X) \otimes H^j(Y) \cong H^k(X \times Y)$$

is an isomorphism of Hodge structures.

We have compatibility with Gysin map:

COROLLARY 10.2.10. *If $f : X \to Y$ is a holomorphic map of compact Kähler manifolds of dimension n and m respectively, the Gysin map is a morphism*

$$H^i(X) \to H^{i+2(m-n)}(Y)(n-m)$$

Exercises

1. Finish the proof of 10.2.3.

2. Let $C^\bullet, F^\bullet$ be as above, and define $E_1^{pq} = H^{p+q}(Gr^p C^\bullet)$ and $d_1 : E_1^{pq} \to E_1^{p+1,q}$ as the connecting map associated to
$$0 \to Gr^p C^\bullet \to F^p C^\bullet / F^{p+2} C^\bullet \to Gr^p C^\bullet \to 0$$
Show that $d_1 = 0$ if $F^\bullet$ is strict. There is in fact a sequence of such maps $d_1, d_2 \ldots$, and strictness is equivalent to the vanishing of all of them.
3. When $C^\bullet = \mathcal{E}^\bullet(X)$ with its Hodge filtration, show that $E_1^{pq} \cong H^q(X, \Omega_X^p)$ and that d_1 is induced by the $\alpha \mapsto \partial\alpha$ on $\bar{\partial}$-cohomology. Conclude that these maps vanish.

10.3. Hodge decomposition for Moishezon manifolds

A compact complex manifold X is called *Moishezon* if its field of meromorphic functions has transcendence degree equal to $\dim X$. This class includes smooth proper algebraic varieties, and forms a natural class in which to do complex algebraic geometry. Moishezon manifolds need not be Kähler, see [**Har**, appendix B] for explicit examples, nevertheless theorem 10.2.4 holds for these manifolds. This is by a result of Moishezon [**Mo**] who proved that such manifolds can always be blown up to a smooth projective varieties and the following.

THEOREM 10.3.1. *If X is a compact complex manifold for which there exists a compact Kähler manifold and a surjective bimeromorphic map $f : \tilde{X} \to X$, then the conclusion of theorem 10.2.4 holds.*

Proofs can be found in [**D1, DGMS**]. We will be content outline a proof. Let $n = \dim X$. There is a map
$$f^* : H^q(X, \Omega_X^p) \to H^q(\tilde{X}, \Omega_{\tilde{X}}^p)$$
which is induced by the map $\alpha \mapsto f^*\alpha$ of (p,q)-forms. We claim that the map f^* is injective. To see this, we need the full strength of Serre duality. There is a pairing
$$H^q(X, \Omega_X^p) \otimes H^{n-q}(X, \Omega_X^{n-p}) \to \mathbb{C}$$
induced by
$$(\alpha, \beta) \to \int_X \alpha \wedge \beta$$

THEOREM 10.3.2 (Cartan). *If X is a compact complex manifold,* $\dim H^q(X, \Omega_X^p) < \infty$.

THEOREM 10.3.3 (Serre). *If X is a compact complex manifold, the pairing described above is perfect*

PROOF. Both results can be deduced from a Hodge decomposition theorem of $\bar{\partial}$, see [**GH**]. □

Therefore, we can define a map
$$f_* : H^q(\tilde{X}, \Omega_{\tilde{X}}^p) \to: H^q(X, \Omega_X^p),$$
analogous to the Gysin map, as the dual to the pullback map
$$H^{n-q}(X, \Omega_X^{n-p}) \to H^{n-q}(\tilde{X}, \Omega_{\tilde{X}}^{n-p})$$
We have $f_* f^*(\alpha) = \alpha$ which proves injectivity of f^* as claimed. By similar reasoning, $f^* H^i(X, \mathbb{C}) \to H^i(\tilde{X}, \mathbb{C})$ is also injective.

Consider the commutative diagram

$$\begin{array}{ccc} H^q(\Omega^p_X) & \xrightarrow{d_1} & H^q(\Omega^{p+1}_X) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ H^q(\Omega^p_{\tilde X}) & \xrightarrow{d_1} & H^q(\Omega^{p+1}_{\tilde X}) \end{array}$$

Where horizontal maps were defined in the previous exercises. Since the bottom d_1 vanishes, the same goes for the top. Although, we have not defined them the same reasoning applies to the higher differentials. This will prove strictness of the Hodge filtration F, and also $\bar{F}$. We can now argue as in the proof of theorem 10.2.4 that the filtrations give a Hodge structure on $H^i(X)$

Exercises

1. Check this version Serre duality for Kähler manifolds using what we have already proven.

10.4. Hypercohomology

It is possible to describe the relationship between the De Rham and Dolbeault cohomologies in more direct terms than we have done so far. But first, we need to generalize the constructions given in chapter 3. Recall that a complex of sheaves is a possibly infinite sequence of sheaves

$$\ldots \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \ldots$$

satisfying $d^{i+1}d^i = 0$. We say that the complex is *bounded* if only finitely many of these sheaves are nonzero. Given any sheaf $\mathcal{F}$ and natural number n, we get a bounded complex $\mathcal{F}[n]$ consisting of $\mathcal{F}$ in the $-n$th position, and zeros elsewhere. The collection of bounded (respectively bounded below) complexes of sheaves on a space X forms a category $C^b(X)$, where a morphism of complexes $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is defined to be a collection of sheaf maps $\mathcal{E}^i \to \mathcal{F}^i$ which commute with the differentials. This category is abelian. We define additive functors $\mathcal{H}^i : C^b(X) \to Ab(X)$ by

$$\mathcal{H}^i(\mathcal{F}^\bullet) = ker(d^i)/image(d^{i-1})$$

A morphism in $C^+(X)$ is a called a *quasi-isomorphism* if it induces isomorphisms on all the sheaves $\mathcal{H}^i$.

Theorem 10.4.1. *Let X be a topological space, then there are additive functors $\mathbb{H}^i : C^+(X) \to Ab$, with $i \in \mathbb{N}$, such that*

1. *For any sheaf $\mathcal{F}$, $\mathbb{H}^i(X, \mathcal{F}[n]) = H^{i+n}(X, \mathcal{F})$*
2. *If $\mathcal{F}^\bullet$ is a complex of acyclic sheaves, $\mathbb{H}^i(X, \mathcal{F}^\bullet) = H^i(\Gamma(X, \mathcal{F}^\bullet)))$.*
3. *If $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism, then the induced map $\mathbb{H}^i(X, \mathcal{E}^\bullet) \to \mathbb{H}^i(X, \mathcal{F}^\bullet)$ is an isomorphism.*
4. *If $0 \to \mathcal{E}^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to 0$ is exact, then there is an exact sequence*

$$0 \to \mathbb{H}^0(X, \mathcal{E}^\bullet) \to \mathbb{H}^0(X, \mathcal{F}^\bullet) \to \mathbb{H}^0(X, \mathcal{G}^\bullet) \to \mathbb{H}^1(X, \mathcal{E}^\bullet) \to \ldots$$

We merely give the construction and indicate the proofs of some of these statements. Complete proofs can be found in [**GM**] or [**I**]. We start by redoing the construction of cohomology for a single sheaf $\mathcal{F}$. The functor G defined in 3.1, gives a flasque sheaf $G(\mathcal{F})$ with an injective map $\mathcal{F} \to G(\mathcal{F})$. $C^1(\mathcal{F})$ is the cokernel of this map. Applying G again yields a sequence

$$\mathcal{F} \to G(\mathcal{F}) \to G(C^1(\mathcal{F}))$$

By continuing as above, we get a resolution by flasque sheaves

$$\mathcal{F} \to G^0(\mathcal{F}) \to G^1(\mathcal{F}) \to \dots$$

Theorem 4.1.3 shows that $H^i(X, \mathcal{F})$ is the cohomology of the complex $\Gamma(X, G^\bullet(\mathcal{F}))$, and this gives the clue how to generalize the construction. The complex $G^\bullet$ is functorial. Given a complex

$$\dots \mathcal{F}^i \xrightarrow{d} \mathcal{F}^{i+1} \dots$$

we get a commutative diagram

$$\begin{array}{ccccc} \dots & \mathcal{F}^i & \xrightarrow{d} & \mathcal{F}^{i+1} \dots \\ & \downarrow & & \downarrow \\ & G^0(\mathcal{F}^i) & \xrightarrow{d} & G^0(\mathcal{F}^{i+1}) \\ & \downarrow \partial & & \downarrow \partial \\ & \dots & & \dots \end{array}$$

We define, the total complex

$$T^i = \bigoplus_{p+q=i} G^p(\mathcal{F}^q)$$

with a differential $\delta = d + (-1)^q \partial$. We can now define

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) = H^i(\Gamma(X, T^\bullet))$$

When applied to $\mathcal{F}[0]$, this yields $H^i(\Gamma(X, G^\bullet(\mathcal{F})))$ which as we have seen is $H^i(X, \mathcal{F})$, and this proves 1 when $n = 0$.

The precise relationship between the various (hyper)cohomology groups is usually expressed by the spectral sequence

$$E_1^{pq} = H^q(X, \mathcal{E}^p) \Rightarrow \mathbb{H}^{p+q}(\mathcal{E}^\bullet)$$

This has the following consequences that we can prove directly:

Corollary 10.4.2. *Suppose that $\mathcal{E}^\bullet$ is a bounded complex of sheaves of vector spaces then*

$$\dim \mathbb{H}^i(\mathcal{E}^\bullet) \le \sum_{p+q=i} \dim H^q(X, \mathcal{E}^p)$$

and

$$\sum (-1)^i \dim \mathbb{H}^i(\mathcal{E}^\bullet) = \sum (-1)^{p+q} \dim H^q(X, \mathcal{E}^p)$$

and

Proof. Suppose that $\mathcal{E}^N$ is the last nonzero term of $\mathcal{E}^\bullet$. Let $\mathcal{F}^\bullet$ be the complex obtained by replacing $\mathcal{E}^N$ by zero in $\mathcal{E}^\bullet$. There is an exact sequence

$$0 \to \mathcal{E}^N[-N] \to \mathcal{E}^\bullet \to \mathcal{F}^\bullet \to 0$$

which induces a long exact sequence of hypercohomology. Then the corollary follows by induction on the length (number of nonzero entries) of $\mathcal{E}^\bullet$. □

Corollary 10.4.3. *Suppose that $\mathcal{E}^\bullet$ is a bounded complex with $H^q(X, \mathcal{E}^p) = 0$ for all $p + q = i$, then $\mathbb{H}^i(\mathcal{E}^\bullet) = 0$.*

In order to facilitate the computation of hypercohomology, we need a criterion for when two complexes are quasi-isomorphic. We will say that a filtration

$$\mathcal{E}^\bullet \supseteq F^p\mathcal{E}^\bullet \supseteq F^{p+1}\mathcal{E}^\bullet \dots$$

which is finite (of length $\leq n$) if $\mathcal{E}^\bullet = F^a\mathcal{E}^\bullet$ and $F^{a+n}\mathcal{E}^\bullet = 0$ for some a.

LEMMA 10.4.4. *Let $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of bounded complexes. Suppose that $F^p\mathcal{E}^\bullet$ and $G^p\mathcal{F}^\bullet$ are finite filtrations by subcomplexes such that $f(F^p\mathcal{E}^\bullet) \subseteq G^p\mathcal{F}^\bullet$. If the induced maps*

$$Gr^p_F(\mathcal{E}^\bullet) \to Gr^p_G(\mathcal{F}^\bullet)$$

are quasi-isomorphisms for all p, then f is a quasi-isomorphism.

Exercises

1. Prove lemma 10.4.4 by induction on the length.

10.5. Holomorphic de Rham complex

To illustrate the ideas from the previous section, let us reprove de Rham's theorem. Let X be a C^∞ manifold. We can resolve $\mathbb{C}_X$ by the complex of C^∞ forms $\mathcal{E}^\bullet_X$, which is acyclic. In other words, $\mathbb{C}_X$ and $\mathcal{E}^\bullet_X$ are quasi-isomorphic. It follows that

$$H^i(X, \mathbb{C}_X) = \mathbb{H}^i(X, \mathbb{C}_X[0]) \cong \mathbb{H}^i(X, \mathcal{E}^\bullet_X) \cong H^i(\Gamma(X, \mathcal{E}^\bullet_X))$$

The last group is just de Rham cohomology.

Now suppose that X is a (not necessarily compact) complex manifold. Then we define a subcomplex

$$F^p\mathcal{E}^\bullet_X = \sum_{p' \geq p} \mathcal{E}^{p'q}_X$$

The image of the map

$$\mathbb{H}^i(X, F^p\mathcal{E}^\bullet_X) \to \mathbb{H}^i(X, \mathcal{E}^\bullet_X)$$

is the filtration introduced just before theorem 10.2.4. We want to reinterpret this purely in terms of holomorphic forms. We define the holomorphic de Rham complex by

$$\mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \dots$$

This has a filtration (sometimes called the "stupid" filtration)

$$\sigma^p\Omega^\bullet_X = \dots 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \dots$$

gotten by dropping the first $p-1$ terms. We have a natural map $\Omega^\bullet_X \to \mathcal{E}^\bullet_X$ which takes σ^p to F^p. Dolbeaut's theorem 8.2.3 implies that F^p/F^{p+1} is quasi-isomorphic to $\sigma^p/\sigma^{p+1} = \Omega^p_X[-p]$. Therefore, lemma 10.4.4 implies that $\Omega^\bullet_X \to \mathcal{E}^\bullet_X$, and more generally $\sigma^p\Omega^\bullet_X \to F^p\mathcal{E}^\bullet_X$, are quasi-isomorphisms. Thus:

LEMMA 10.5.1. *$H^i(X, \mathbb{C}) \cong \mathbb{H}^i(X, \Omega^\bullet_X)$ and $F^pH^i(X, \mathbb{C})$ is the image of $\mathbb{H}^i(X, \sigma^p\Omega^\bullet_X)$.*

When X is compact Kähler, theorem 10.2.4 implies that the map

$$\mathbb{H}^i(X, \sigma^p\Omega^\bullet_X) \to \mathbb{H}^i(X, \Omega^\bullet_X)$$

is injective.

From corollaries 10.4.2 and 10.4.3, we obtain

Corollary 10.5.2. *If X is compact, the ith Betti number*

$$b_i(X) \leq \sum_{p+q=i} \dim H^q(X, \Omega^p_X)$$

and the Euler characteristic

$$e(X) = \sum (-1)^i b_i(X) = \sum (-1)^{p+q} \dim H^q(X, \Omega^p_X)$$

Corollary 10.5.3. *If $H^q(X, \Omega^p_X) = 0$ for all $p + q = i$, then $H^i(X, \mathbb{C}) = 0$.*

Corollary 10.5.4. *Let X be a Stein manifold (see section 14.5), then $H^i(X, \mathbb{C}) = 0$ for $i > \dim X$.*

Proof. This follows from theorem 14.2.10. □

CHAPTER 11

Algebraic Surfaces

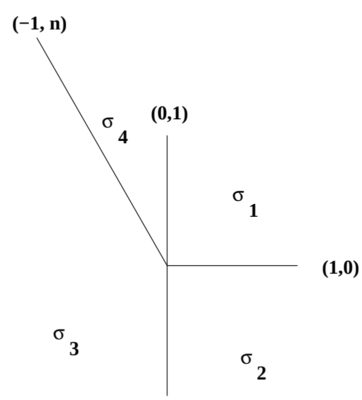
A (nonsingular) complex surface is a two dimensional complex manifold. By an algebraic surface, we will mean a two dimensional smooth projective surface. To be consistent, Riemann surfaces will be referred to as complex curves from now on.

11.1. Examples

Let X be an algebraic surface. We know from the previous chapter that there are three interesting Hodge numbers h^{10}, h^{20}, h^{11}. The first two are traditionally called (and denoted by) the *irregularity* ($q = h^{10}$) and *geometric genus* ($p_g = h^{20}$).

EXAMPLE 11.1.1. *If $X = \mathbb{P}^2$, then $q = p_g = 0$ and $h^{11} = 1$. In fact, as Hodge structures $H^1(\mathbb{P}^2) = 0$ and $H^2(\mathbb{P}^2) = \mathbb{Z}(-1)$.*

EXAMPLE 11.1.2. *The next example is the rational ruled surface F_n, with n a nonnegative integer. This can be conveniently described as the toric variety (section 2.5) associated to the fan*



These surfaces are not isomorphic for different n. However they have the same invariants $q = p_g = 0$ and $h^{11} = 2$.

EXAMPLE 11.1.3. *If $X = C_1 \times C_2$ is a product of two nonsingular curves of genus g_1 and g_2. Then by Künneth's formula,*

$$H^1(X) \cong H^1(C_1) \oplus H^1(C_2)$$

$$H^2(X) \cong [H^2(C_1) \otimes H^0(C_2)] \oplus [H^1(C_1) \otimes H^1(C_2)] \oplus [H^0(C_1) \otimes H^2(C_2)]$$

as Hodge structures. Therefore $q = g_1 + g_2$, $p_g = g_1 g_2$ and $h^{11} = 2g_1g_2 + 2$.

EXAMPLE 11.1.4. *Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. Then $q = 0$. We will list the first few values:*

d	p_g	h^{11}
2	*0*	*2*
3	*0*	*7*
4	*1*	*20*
5	*4*	*45*
6	*10*	*86*

These can be calculated using formulas given later (15.2.5).

Example 11.1.5. *An elliptic surface is surface X which admit surjective morphism $f : X \to C$ to a smooth projective curve such that all nonsingular fibers are elliptic curves. We will see some examples in the next chapter along with a calculation of q.*

A method of generating new examples from old is by blowing up. The basic construction is: Let

$$Bl_0\mathbb{C}^2 = \{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \,|\, x \in \ell\}$$

The projection $p_1 : Bl_0\mathbb{C}^2 \to \mathbb{C}^2$ is birational. This is called the blow up of $\mathbb{C}^2$ at 0. This can be generalized to yield the blow up $Bl_pX \to X$ of the surface X at point p. This is an algebraic surface which can be described analytically as follows. Let $B \subset X$ be a coordinate ball centered at p. After identifying B with a ball in $\mathbb{C}^2$ centered at 0, we can form let Bl_0B be the preimage of B in $Bl_0\mathbb{C}^2$. The boundary of Bl_0B can be identified with the boundary of B. Thus we can glue $X - B \cup Bl_0B$ to form Bl_pX.

Using the above description, we can compute $H^*(Y, \mathbb{Z})$ where $Y = Bl_pX$ by comparing Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\to & H^i(X) & \to & H^i(X - B') & \oplus & H^i(B) & \to & H^i(X - B' \cap B) \\
 & \downarrow & & || & \oplus & \downarrow & & || \\
\to & H^i(Y) & \to & H^i(Y - Bl_pB') & \oplus & H^i(Bl_pB) & \to & H^i(Y - Bl_pB' \cap Bl_pB)
\end{array}$$

where $B' \subset B$ is a smaller ball. It follows that

Lemma 11.1.6. *$H^1(Y) \cong H^1(X)$ and $H^2(Y) = H^2(X) \oplus \mathbb{Z}$.*

Corollary 11.1.7. *q and p_g are invariant under blowing up. $h^{11}(Y) = h^{11}(X) + 1$.*

Proof. The lemma implies that $b_1 = 2q$ is invariant and $b_2(Y) = b_1(X) + 1$. Since $b_2 = 2p_g + h^{11}$, the only possiblities are $h^{11}(Y) = h^{11}(X) + 1$, $p_g(Y) = p_g(X)$, or that $p_g(Y) < p_g(X)$. The last inequallity means that there is a nonzero holomorphic 2-form on X that vanishes on $X - p$, but this is impossible. □

A morphism of varieties $f : V \to W$ is called birational if it is an isomorphism when restricted to nonempty Zariski open sets of X and Y. A birational map $V \dashrightarrow W$ is simply an isomorphism of Zariski open sets. Blow ups and their inverses ("blow downs") are examples of birational morphisms and maps. For the record, we point out

Theorem 11.1.8 (Castelnuovo). *Any birational map between algebraic surfaces is a given by a finite sequence of blow ups and downs.*

Therefore, we get the birational invariance of q and p_g. However, there are easier ways to prove this. Blow ups can also be used to extend meromorphic functions. A meromorphic function $f : X \dashrightarrow \mathbb{P}^1$ is a holomorphic map from an open subset $U \subseteq \mathbb{P}^1$ which can be expressed locally as a ratio of holomorphic functions.

THEOREM 11.1.9 (Zariski). *If $f : X \dashrightarrow \mathbb{P}^1$ meromorphic function on an algebraic surface. Then there is a finite sequence of blow ups such that $Y \to X$ such that f extends to a holomorphic map $f' : X \to \mathbb{P}^1$.*

Exercises

1. Compare the construction of the blow up given here and in section 2.5 (identify $((x,t)$ with $((x,xt),[t,1]))$.
2. Finish the proof of lemma 11.1.6.

11.2. Castenuovo-de Franchis' theorem

When can try to study varieties by mapping them onto lower dimensional varieties. In the case of surfaces, the target should be a curve. A very useful criterion for this is

THEOREM 11.2.1 (Castenuovo-de Franchis). *Suppose X is an algebraic surface. A necessary and sufficient condition for X to admit a constant holomorphic map to a smooth curve of genus $g \geq 2$ is that there exists two linear independant forms $\omega_i \in H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \omega_2 = 0$.*

PROOF. The necessity is clear. We will sketch the converse, since it gives a nice application of corollary 10.2.6. A complete proof can be found in [**BPV**, pp123-125]. Choosing local coordinates, we can write

$$\omega_i = f_i(z_1, z_2)dz_1 + g_i(z_1, z_2)dz_2$$

The condtion $\omega_1 \wedge \omega_2 = 0$ is

$$(f_1 g_2 - f_2 g_1)dz_1 \wedge dz_2 = 0$$

which implies that

$$f_2/f_1 = g_2/g_1 = F$$

Thus $\omega_2 = F\omega_1$. Since ω_i are globally defined, $F = \omega_2/\omega_1$ defines a global meromorphic function $X \dashrightarrow \mathbb{P}^1$. By theorem 11.1.9, there exists a blow up $Y \to X$ such that F extends to a holomorphic function $F' : Y \to \mathbb{P}^1$. The fibers of F' will not be connected. Stein's factorization theorem [**Har**] shows that the map can be factored as

$$Y \xrightarrow{\Phi} C \to \mathbb{P}^1,$$

where Φ has connected fibers and $C \to \mathbb{P}^1$ is a finite to one map of smooth projective curves. To avoid too much notation, let us denote the pullbacks of ω_i to Y by ω_i as well.

We claim that ω_i are pullback from holomorphic 1-forms on C. We will check this locally around a general point $p \in X$. Since ω_i are closed (corollary 10.2.6),

$$d\Phi \wedge \omega_1 = d\omega_2 = 0 \tag{23}$$

Let t_1 be a local coordinate centered at $\Phi(p)$. Let us also denote the pullback of this function to neighbbourhood of p by t_1. Then we can choose a function t_2 such that t_1, t_2 give local coordinates at p. (23) becomes $dt_1 \wedge \omega_1 = 0$. Consequently $\omega_1 = f(t_1, t_2)dt_1$, for some function f. The relation $d\omega_1 = 0$, implies f is a function of t_1 alone. Thus ω_1 is locally the pullback of a 1-form on C as claimed. The same reasoning applies to ω_2. The final step is to prove that, the blow up is unnecessary, i.e. we can take $Y = X$. □

An obvious corollary is:

COROLLARY 11.2.2. *If $q \geq 2$ and $p_g = 0$, then X admits a constant map to a curve as above.*

This can be improved substantially [**BPV**, IV, 4.2].

11.3. The Neron-Severi group

Let X be an algebraic surface once again. The image of the first Chern class map is the Neron-Severi group $NS(X) = H^2(X,\mathbb{Z}) \cap H^{11}(X)$. The rank of this group is called Picard number $\rho(X)$. We have $\rho \leq h^{11}$ with equality if $p_g = 0$.

A divisor on X is a finite integer linear combination $\sum n_i D_i$ of possibly singular irreducible curves $D_i \subset X$. We can define a line bundle $\mathcal{O}_X(D)$ as we did for Riemann surfaces in section 5.6. If f_i are local equations of $D_i \cap U$ in some open set U,

$$\mathcal{O}_X(D)(U) = \mathcal{O}_X(U)\frac{1}{f_1^{n_1} f_2^{n_2} \cdots}$$

is a fractional ideal. In particular, if D is irreducible $\mathcal{O}_X(-D)$ is the ideal sheaf of D.

LEMMA 11.3.1. *If D_i are smooth curves, then $c_1(\mathcal{O}_X(\sum n_i D_i)) = \sum n_i [D_i]$.*

PROOF. This is an immediate consequence of theorem 6.5.6. □

The cup product pairing

$$H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) \to \mathbb{Z}$$

restricts to a pairing on $NS(X)$ denoted by "$\cdot$".

LEMMA 11.3.2. *Given a pair of transverse smooth curves D and E,*

$$D \cdot E = \int_X c_1(\mathcal{O}(D)) \cup c_1(\mathcal{O}(E)) = \#(D \cap E)$$

PROOF. This follows from the lemma 11.3.1 and proposition 4.4.3. The number $i_p(D,E)$ is seen to be always $+1$ in this case. □

If the intersection of the curves D and E is finite but not transverse, it is still possible to give a geometric meaning to the above product. For each $p \in X$, let $\mathcal{O}_p^{an}$ and $\mathcal{O}_p$ be the rings of germs of holomorphic respectively regular functions at p. Let $f, g \in \mathcal{O}_p \subset \mathcal{O}_p^{an}$ be the local equations of D and E respectively. Define $i_p(D,E) = \dim \mathcal{O}_p/(f,g)$. Then

LEMMA 11.3.3. *If we identify $\mathcal{O}_p^{an}$ with the ring of convergent power series in two variables.*

$$i_p(D,E) = \dim \mathcal{O}_p^{an}/(f,g) = \frac{1}{(2\pi\sqrt{-1})^2}\int_{|f(z)|=|g(z)|=\epsilon} \frac{df \wedge dg}{fg}$$

PROOF. [**GH**, pp. 662-670]. □

The integral above has a pretty interpretation as the linking number[1] of the knots $S^3 \cap \{f = 0\}$ and $S^3 \cap \{g = 0\}$, where S^3 is a small sphere about $0 \in \mathbb{C}^2$.

PROPOSITION 11.3.4. *If D, E are smooth curves such $D \cap E$ is finite, then*

$$D \cdot E = \sum_{p \in X} i_p(D, E)$$

Recall that $H^2(\mathbb{P}^2, \mathbb{Z}) = H^4(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$, and the generator of $H^2(\mathbb{P}^2, \mathbb{Z})$ is the class of line $[L]$. Given a curve D defined by a polynomial f, we define $\deg D = \deg f$. Then $[D] = \deg D[L]$.

COROLLARY 11.3.5 (Bezout). *If D, E are smooth curves on $\mathbb{P}^2$ with a finite intersection,*

$$\sum_{p \in X} i_p(D, E) = \deg(D)\deg(E)$$

PROOF. We have $[D] = \deg D[L]$, and $[E] = \deg E[L]$, so that $D \cdot E = \deg D \deg E L \cdot L = \deg D \deg E$. □

Given a surjective morphism $f : X \to Y$ of algebraic surfaces. The pullback $f* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ preserves the Neron-Severi group and the intersection pairing. This can be interpreted directly in the language of divisors. Given an irreducible divisor D on Y. We can make the set theoretic preimage $f^{-1}D$ into a divisor by pulling back the ideal, i.e. $\mathcal{O}_X(-f^{-1}D) = \mathcal{O}_Y(-D)$. We extend this operation to all divisors by linearity. The operation satisfies $f^*\mathcal{O}_Y(D) = \mathcal{O}_X(f^{-1}D)$. Since c_1 is functorial, we get

LEMMA 11.3.6. *f^{-1} is compatible with f^* on $NS(X)$.*

Exercises

1. Let $f : X \to C$ be a morphism from a surface to a curve. The fiber over $p \in C$ is the set $f^{-1}(p)$ made into a divisor using the ideal $f^{-1}\mathcal{O}_C(-p)$. Show that any two fibers determine the same class in $NS(X)$. Conclude that $F^2 = 0$ for any fiber F.
2. Let $X = C \times C$ be product of a curve with itself. Consider, the divisors $H = C \times \{p\}$, $V = \{p\} \times C$ and the diagonal Δ. Compute their intersection numbers, and show that these are independent in $NS(X) \otimes \mathbb{Q}$.
3. Let $\pi : Y \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at point. Show that Y admits a natural morphism to $\mathbb{P}^1$, and let F be the fiber. Let L be the pullback of a line under π. These give a basis for $NS(Y)$. Compute the intersection numbers.
4. A divisor is called (very) ample if $\mathcal{O}_X(H)$ is. If H is ample, then prove $H^2 > 0$. Show that converse fails.
5. $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve, and let $X = E \times E$. Calculate the Picard number, and show that it could be 3 or 4 depending on τ.

[1] Dip one of the knots in soap solution, and count the number of times the other intersects the bubble.

11.4. The Hodge index theorem

Let X a compact Kähler surface. Then the Kähler form ω is closed real $(1,1)$ form. Therefore the Kähler class $[\omega]$ is an element of $H^{11}(X) \cap H^2(X, \mathbb{R})$.

THEOREM 11.4.1. *Let X be a compact Kähler surface. Then the restriction of the cup product to $(H^{11}(X) \cap H^2(X, \mathbb{R})) \cap (\mathbb{R}[\omega])^{\perp}$ is negative definite.*

PROOF. Locally, we can find an orthonormal basis $\{\phi_1, \phi_2\}$ of $\mathcal{E}^{(1,0)}$. In this basis,

$$\omega = \frac{\sqrt{-1}}{2}(\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2)$$

and the volume form

$$dvol = \frac{\omega^2}{2} = \left(\frac{\sqrt{-1}}{2}\right)^2 \phi_1 \wedge \bar{\phi}_1 \wedge \phi_2 \wedge \bar{\phi}_2$$

using the exercises from 9.2. Choose an element of $(H^{11}(X) \cap H^2(X, \mathbb{R}))$ and represent it by a harmonic real $(1,1)$ form, then

$$\alpha = \sum a_{ij} \phi_i \wedge \bar{\phi}_j$$

with $a_{ji} = \bar{a}_{ij}$. If α is also chosen in $(\mathbb{R}[\omega])^{\perp}$, then

$$\alpha \wedge \omega = \frac{2}{\sqrt{-1}}(a_{11} + a_{22}) dvol$$

is exact. This is impossible unless $a_{11} + a_{22} = 0$. Therefore

$$\int_X \alpha \wedge \alpha = 2\left(\frac{2}{\sqrt{-1}}\right)^2 \int_X (|a_{11}|^2 + |a_{12}|^2) dvol < 0$$

□

COROLLARY 11.4.2. *If H is an ample divisor on an algebraic surface, the intersection pairing is negative definite on $(NS(X) \otimes \mathbb{R}) \cap (\mathbb{R}[H])^{\perp}$*

PROOF. By corollary 9.1.8, $[H]$ is a Kähler class. □

Therefore when the cup product form is diagonalized, the diagonal consists of one 1 followed by $\rho - 1$ -1's. So we obtain

COROLLARY 11.4.3. *If H, D are divisors on an algebraic surface such that $H^2 > 0$ and $D \cdot H = 0$, then $D^2 < 0$*

Exercises

1. Prove that the restriction of the cup product to $(H^{20}(X) + H^{02}(X)) \cap H^2(X, \mathbb{R})$ is positive definite.
2. Conclude that (the matrix representing) the cup product pairing has $2p_g+1$ positive eigenvalues. Therefore p_g is a topological invariant.
3. Let $f : X \to Y$ be a morphism from a smooth algebraic surface to a possibly singular projective surface. Consider the set $\{D_i\}$ of irreducible curves which map to points under f. Prove a theorem of Mumford that the matrix $(D_i \cdot D_j)$ is negative definite.

CHAPTER 12

Topology of families

12.1. Fiber bundles

A C^∞ map $f : X \to Y$ of manifolds is called a *fiber bundle* if it is locally a product of Y with another manifold F (called the fiber). This means that there is an open cover $\{U_i\}$ and diffeomorphisms $f^{-1}U_i \cong U_i \times F$ compatible with the projections. If $X = Y \times F$, the bundle is called trivial. Nontrivial bundles over S^1 can be constructed as follows. Let F be a manifold with a diffeomorphism $\phi : F \to F$, then glue $F \times \{0\}$ in $F \times [0,1]$ to $F \times \{1\}$ by identifying $(x, 0)$ to $(\phi(x), 1)$. This includes the familiar example of the Mobius strip where $F = \mathbb{R}$ and ϕ is multiplication by -1. If the induced map $\phi^* : H^*(F) \to H^*(F)$, called the *monodromy transformation*, is nontrivial then the fiber bundle is nontrivial.

A C^∞ map $f : X \to Y$ is called a *submersion* if the map on tangent spaces is surjective. The fibers of such a map are submanifolds. A continuous map of topological spaces is called proper if the preimage of any compact set is compact.

Theorem 12.1.1 (Ehresmann). *Let $f : X \to Y$ be a proper submersion of C^∞ connected manifolds. Then f is a C^∞ fiber bundle; in particular, the fibers are diffeomorphic.*

Proof. A complete proof can be found in [**MK**, 4.1]. Here we sketch the proof of the last statement. Since Y is connected, we can join any two points, say 0 and 1 by a path. Thus we replace Y by $\mathbb{R}$. Choose a Riemmanian metric on X. The gradient ∇f is defined as the vector field dual to *df* under the inner product associated to the metric. By assumption *df* and therefore ∇f is nowhere zero. The existence and uniqueness theorem of ordinary linear differential equations allows us to define, for each $p \in f^{-1}(0)$, a C^∞ path $\gamma_p : [0,1] \to X$ passing through p at time 0 and with velocity ∇f. Then the gradient flow $p \mapsto \gamma_p(1)$ gives the desired diffeomorphism. □

Exercises

1. Show that theorem can fail for nonproper maps

12.2. Some elliptic surfaces

Let's look some complex analytic examples. Let Γ be the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}$ where $a \in \mathbb{Z}$ acts by the matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. More explicitly, the elements are triples $(m, n, a) \in \mathbb{Z}^3$ with multiplication

$$(m, n, a) \cdot (m', n', a') = (m + m' + an', n + n', a + a')$$

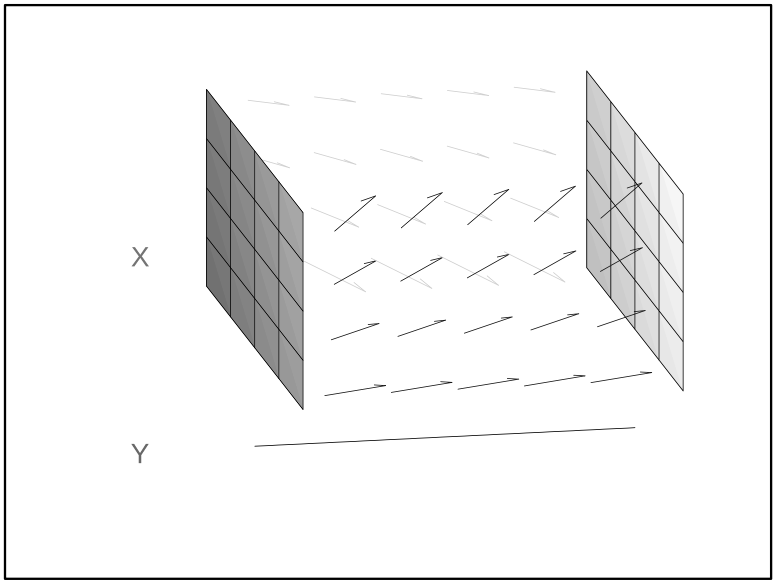


FIGURE 1. Gradient flow

Let Γ act on $\mathbb{C} \times H$ by

$$(m, n, a) : (z, t) \mapsto (z + m + n\tau, \tau + a)$$

The quotient gives a holomorphic family of elliptic curves

$$\mathcal{E} \to H/\mathbb{Z} \overset{\tau \to q = \exp(-2\pi i \tau)}{\cong} D^*$$

over the punctured disk. As a C^∞ map, it is a fiber bundle with a torus T as fiber. However, it is not a locally holomorphic fiber bundle. The fiber over τ, $\mathcal{E}_\tau$ is given by the Weirstrass equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$$

This equation makes sense in the limit as $\tau \to \infty$ (or equivalently as $q \to 0$), and it defines a rational curve with a node. In this, way $\mathcal{E} \to D^*$ can be completed to a map of complex manifolds $\bar{\mathcal{E}} \to D$. We have $\mathcal{E}_\tau \cong \mathbb{C}/(\tau^{-1}\mathbb{Z} + \mathbb{Z})$. Let $a(\tau)$ be the image in $\mathcal{E}_\tau$ of the line segment joining 0 to 1, and let $b(\tau)$ be the image of the segment joining 0 to τ^{-1}. If we orient these so that $a \cdot b = 1$, these form a basis of $H_1(\mathcal{E}_t, \mathbb{Z})$. $b(t)$ is called a *vanishing cycle* since it shrinks to the node as $t \to 0$, see figure 2

The restriction of $\mathcal{E}$ to a bundle over the circle $S = \{t \mid |t| = \epsilon\}$ is a fiber bundle with monodromy transformation μ given by Picard-Lefschetz formula:

$$\mu(a) = a + b, \ \mu(b) = b.$$

From a more topological point of view, $\mathcal{E}|_S$ can be described by taking a trivial torus bundle $T \times [0, 1]$ and gluing the ends $T \times \{0\}$ and together $T \times \{1\}$ using a so called Dehn twist about the vanishing cycle $b = b(t)$. This is a diffeomorphism

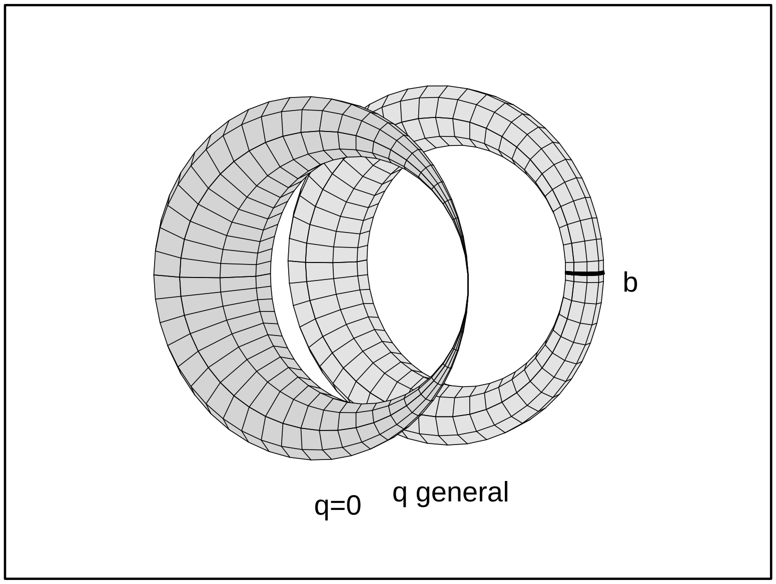


FIGURE 2. Vanishing cycle

which is the identity outside a neighbourhood U of b and twists "once around" along b (see figures 3 and 3, U is the shaded region).

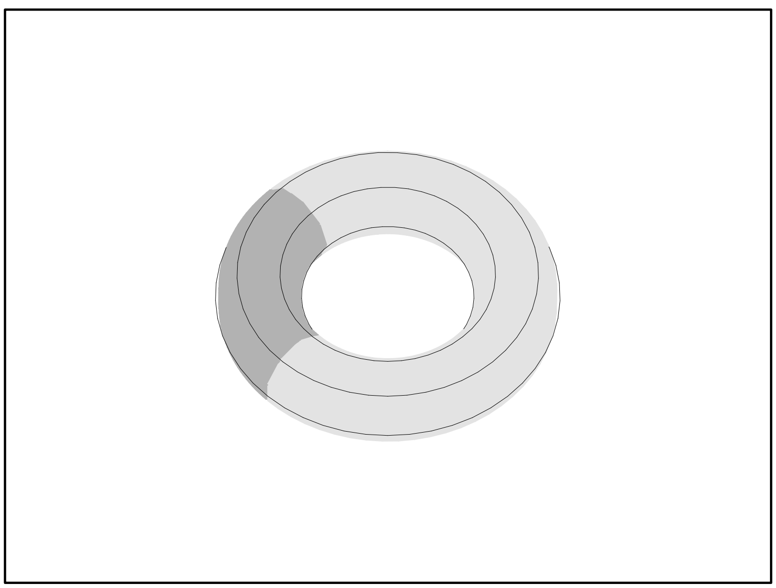


FIGURE 3. T foliated by meridians

Next, consider the family of elliptic curves in Legendre form

$$\mathcal{E} = \{([x,y,z],t) \in \mathbb{P}^2 \times \mathbb{C} - \{0,1\} \mid y^2z - x(x-z)(x-tz) = 0\} \to \mathbb{C}$$

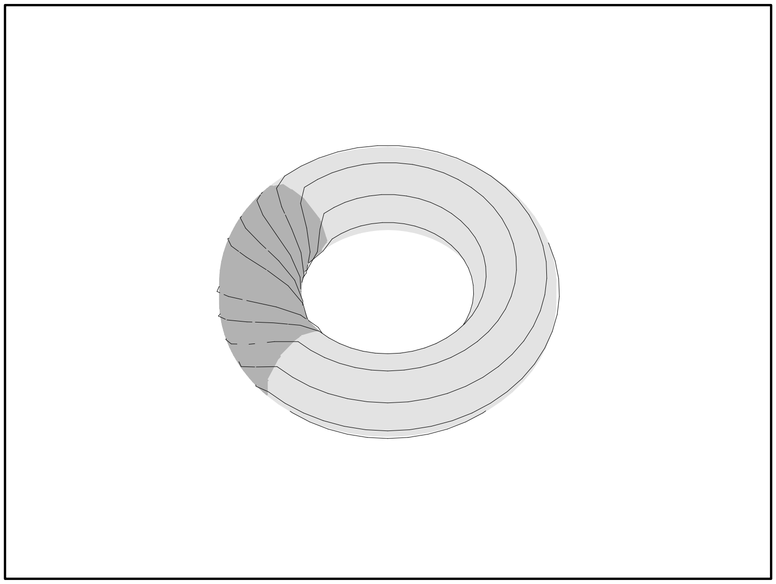


FIGURE 4. T foliated by images of meridians under Dehn twist

Let's see how to calculate the monodromy of going around 0 and 1. The above equation is meaniningful if $t = 0, 1$, and it defines a rational curve with a single node. By introducing $s = t^{-1}$, we get an equation

$$sy^2z - x(x-z)(sx-z) = 0$$

which defines a union of lines when $s = 0$. In this way, we can extend $\mathcal{E}$ to a surface $\mathcal{E}' \to \mathbb{P}^1$. Unfortunately, $\mathcal{E}'$ is singular, and it is necessary to resolve singularities to get an nonsingular surface $\bar{\mathcal{E}}$ containing $\mathcal{E}$ (we can take the minimal desingularization, which for our purposes means that $b_2(\bar{\mathcal{E}})$ is chosen as small as possible). In the process, the nodal curves get replaced by the more complicated fibers, so the Picard-Lefschetz formula will not apply. However, it is possible to calculate this from a different point of view. Recall that $H/\Gamma(2) = \mathbb{P}^1 - \{0, 1, \infty\}$. $\mathcal{E}$ can also be realized as a quotient of $\mathbb{C} \times H$ by an action of the semidirect product $\mathbb{Z}^2 \rtimes \Gamma(2)$ as above. The $\Gamma(2)$ action extends to $H^* = H \cup \mathbb{Q} \cup \{\infty\}$. We can choose a fundamental domain for $\Gamma(2)$ in H^* as depicted in figure 5, the three cusps are $0, 1, \infty$.

The subgroup of $\Gamma(2)$ which fixes ∞ is generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and it follows easily that this is the monodromy matrix for it. We will call $\bar{\mathcal{E}}$ the elliptic modular surface of level two.

Exercises

1. Calculate the monodromy matrices for the remaining cusps in the last example.

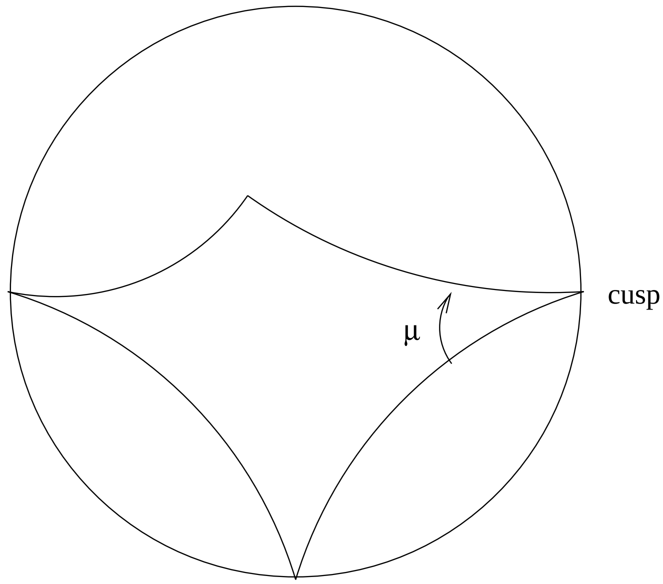


FIGURE 5. Fundamental domain of $\Gamma(2)$

12.3. Local systems

In this section, we give a more formal treatment of monodromy.

Let X be a topological space, a path from $x \in X$ to $y \in X$ is a is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Two paths γ, η are homotopic if there is a continuous map $\Gamma : [0,1] \times [0,1] \to X$ such that $\gamma(t) = \Gamma(t,0)$, $\eta(t) = \Gamma(t,1)$ $\Gamma(0,s) = x$ and $\Gamma(1,s) = y$. We can compose paths: If γ is a path from x to y and η is a path from y to z, then $\gamma \cdot \eta$ is the path given by following one by the other at twice the speed. More formally,

$$\gamma \cdot \eta(t) = \begin{cases} \gamma(2t) & \text{if } t \leq 1/2 \\ \eta(2t-1) & \text{if } t > 1/2 \end{cases}$$

This operation is compatible with homotopy in the obvious sense, and the induced operation on homotopy classes is associative. It almost a group law. To make this precise, we define a category $\Pi(X)$ whose objects are points of X and whose morphisms are homotopy classes of paths. This makes $\Pi(X)$ into a *groupoid* which means that every morphism is an isomorphism. In other words every (homotopy class of a) path has a inverse. This is not a group because it is not possible to compose any two paths. To get around this, we can consider loops, i. e. paths which start and end at the same place. Let $\pi_1(X, x)$ be the set of homotopy classes of loops based (starting and ending)at x. This is just $Hom_{\Pi(X)}(x, x)$ and as such it inherits a composition law which makes it a group called the fundamental group of (X, x). We summarize the standard properties that can be found in almost any toplogy textbook, e.g. [**Sp**]:

1. π_1 is a functor on the category of path connected spaces with base point and base point preserving continous maps.

2. If X is path connected, $\pi_1(X,x) \cong \pi_1(X,y)$ (consequently, we usually suppress the base point).
3. Two homotopy equivalent path connected spaces have isomorphic fundamental groups.
4. Van Kampen's theorem: If X is a path connected simplicial complex which is the union two subcomplexes $X_1 \cup X_2$, then $\pi_1(X)$ is the free product of $\pi_1(X_1)$ and $\pi_1(X_2)$ amalgamated along the image of $\pi_1(X_1 \cup X_2)$.
5. A X is a connected locally path connected space has a universal cover $\pi : \tilde{X} \to X$, and $\pi_1(X)$ is isomorphic to the group of deck transformations, i.e. selfhomeomorphisms of $\tilde{X}$ which commute with π.

This already suffices to calculate the examples of interest to us. It is easy to see that the fundmental group of the circle $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Z}$. The complement in $\mathbb{C}$ of a set S of k points is homotopic to a wedge of k circles. Therefore $\pi_1(\mathbb{C} - S)$ is a free group on k generators.

Let X be a topological space. A *local system* of abelian groups is a functor $F : \Pi(X) \to Ab$. A local system F gives rise to a $\pi_1(X,x)$-modules i.e. abelian group $F(x)$ with a $\pi_1(X,x)$ action. We can also define a sheaf $\mathcal{F}$ as follows

$$\mathcal{F}(U) = \{f : U \to \cup F(x) \mid f(x) \in F(x) \text{ and } f(\gamma(1)) = F(\gamma)(f(\gamma(0)))\}$$

This sheaf is *locally constant* which means that every $x \in X$ has an open neighbourhood U such that $\mathcal{F}|_U$ is constant.

Theorem 12.3.1. *Let X be a connected good (i.e. locally path connected semilocally simply connected) topological space. There is an equivalence of categories between*

1. *The category of $\pi_1(X)$-modules.*
2. *The category of local systems and natural transformations.*
3. *The full subcategory of $Sh(X)$ of locally constant sheaves on X.*

Proof. [**I**, IV,9] or [**Sp**, p. 360]. □

In view of this theorem, we will treat local systems and locally constant sheaves as the same.

Let $f : X \to Y$ be a fiber bundle. We are going to construct a local system which takes $y \to H^i(X_y, \mathbb{Z})$. Given a path $\gamma : [0,1] \to Y$ joining y_0 and y_1, the pullback $\gamma^* X = \{(x,t) \mid f(x) = \gamma(t)\}$ will be a trivial bundle over $[0,1]$. Therefore $\gamma^* X$ will deformation retract onto both X_{y_0} and X_{y_1}, and so we have isomorphisms

$$H^i(X_{y_0}) \xleftarrow{\sim} H^i(\gamma^* X) \xrightarrow{\sim} H^i(X_{y_1})$$

The map $H^i(X_{y_0}) \to H^i(X_{y_1})$ can be seen to depend only on the homotopy class of the path, thus we have a local system which gives rise to a locally constant sheaf which will be constructed directly in the next section.

12.4. Higher direct images

In this section, we return to general sheaf theory. Let $f : X \to Y$ be a continuous map and $\mathcal{F} \in Sh(X)$ a sheaf. We can define the higher direct images by imitating the definition of $H^i(X,\mathcal{F})$ in section 3.2.

$$\begin{aligned} R^0 f_*\mathcal{F} &= f_*\mathcal{F} \\ R^1 f_*\mathcal{F} &= coker[f_*G(\mathcal{F}) \to f_*C^1(\mathcal{F})] \\ R^{n+1} f_*\mathcal{F} &= R^1 f_* C^n(\mathcal{F}) \end{aligned}$$

Note that when Y is a point, $Rf^i_*\mathcal{F}$ is just $H^i(X,\mathcal{F})$ viewed as sheaf on it. We have an analogue of theorem 3.2.2.

THEOREM 12.4.1. *Given an exact sequence of sheaves*

$$0 \to A \to B \to C \to 0,$$

there is a long exact sequence

$$0 \to R^0 f_* A \to R^0 f_* B \to R^0 f_* C \to R^1 f_* A \ldots$$

There is an alternative description which is a bit more convenient.

LEMMA 12.4.2. *If $f : X \to Y$ is a continuous map, and $\mathcal{F} \in Sh(X)$, then $R^i f_*\mathcal{F}(U)$ is the sheafification of the presheaf $U \mapsto H^i(U,\mathcal{F})$*

PROOF. Let Γ^i denote the presheaf $U \mapsto H^i(U,\mathcal{F})$. For $i = 0$ there is no need to sheafify, since $f_*\mathcal{F} = \Gamma^0$ by definition of f_*. By our original construction of $H^1(U,\mathcal{F})$, we have an exact sequence

$$f_*G(\mathcal{F})(U) \to f_*C^1(\mathcal{F})(U) \to \Gamma^1(U) \to 0$$

This shows that Γ^1 is the cokernel $f_*G(\mathcal{F}) \to f_*C^1(\mathcal{F})$ in the category of presheaves. Hence $(\Gamma^1)^+ = R^1 f_*\mathcal{F}$. The rest follows by induction □

Each element of $H^i(X,\mathcal{F})$ determines an a global section of the presheaf Γ^i and hence of the sheaf $R^i f_*\mathcal{F}$. This map $H^i(X,\mathcal{F}) \to H^0(X, R^i f_*\mathcal{F})$ is often called an *edge homomorphism*.

Let us now assume that $f : X \to Y$ is a fiber bundle of triangulable spaces. Then choosing a contractible neighbourhood U of y, we see that $H^i(U,\mathbb{Z}) \cong H^i(X_y,\mathbb{Z})$. Since such neigbourhoods are cofinal, it follows that $R^i f_*\mathbb{Z}$ is locally constant. This coincides with the sheaf associated to the local system $H^i(X_y,\mathbb{Z})$ constructed in the previous section. The global sections of $H^0(Y, R^i f_*\mathbb{Z})$ is the space $H^i(X_y,\mathbb{Z})^{\pi_1(Y,y)}$ of invariant cohomology classes on the fiber. We can construct elements of this space using the edge homomorphism. For more general maps, we still have $(R^i f_*\mathbb{Z})_y \cong H^i(X_y,\mathbb{Z})$, but the ranks can jump, so these need not be locally constant.

The importance of the higher direct images is that it allows the cohomology of X to be computed on Y. Let us briefly see how this works. Construct $G^\bullet(\mathcal{F})$ as in section 10.4. Then as we saw $\Gamma(G^\bullet(\mathcal{F}))$ is a complex whose cohomology groups are exactly $H^*(X,\mathcal{F})$. Instead of constructing this in one step, let's consider the complex of sheaves $f_*G^\bullet(\mathcal{F})$ on Y, which we will denote by $\mathbb{R}f_*\mathcal{F}$ even though this isn't technically quite correct[1]. The interesting features of this complex can be summarized by

PROPOSITION 12.4.3.
1. *The ith cohomolgy sheaf $\mathcal{H}^i(\mathbb{R}f_*\mathcal{F}) \cong R^i f_*\mathcal{F}$.*
2. $\mathbb{H}^i(\mathbb{R}f_*\mathcal{F}) \cong H^i(X,\mathcal{F})$

[1] A better aproximation of its true meaning would be to consider it as the quasi-isomorphism class of $f_*G^\bullet(\mathcal{F})$.

To make the relationship between $\mathbb{R}f_*\mathcal{F}$ and $R^i f_*\mathcal{F}$ clearer, let us introduce the truncation operators. For any complex $C^\bullet$ of sheaves (or elements of an abelian category), we can introduce the subcomplex

$$\tau_{\leq p}C^i = \begin{cases} C^p & \text{if } i < p \\ ker(C^p \to C^{p+1}) & \text{if } i = p \\ 0 & \text{otherwise} \end{cases}$$

Truncation yields an increasing filtration. The key property is:

Lemma 12.4.4. *There is an exact sequence of complexes*

$$0 \to \tau_{\leq q-1}C^\bullet \to \tau_{\leq q}C^\bullet \to \mathcal{H}^q(C^\bullet)[-q] \to 0$$

for each p.

Corollary 12.4.5.

$$0 \to \tau_{\leq q-1}\mathbb{R}f_*\mathcal{F} \to \tau_{\leq q}\mathbb{R}f_*\mathcal{F} \to R^q f_*\mathcal{F}[-q] \to 0$$

for each p.

From here it is a straight forward matter to construct the Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

but we won't (but see exercises). However, we will explain one of its consequences.

Proposition 12.4.6. *If $\mathcal{F}$ is a sheaf of vector spaces over a field, such that $\sum \dim H^p(Y, R^q f_*\mathcal{F}) < \infty$, then*

$$\dim H^i(X, \mathcal{F}) \leq \sum_{p+q=i} \dim H^p(Y, R^q f_*\mathcal{F})$$

and

$$\sum \dim(-1)^i H^i(X, \mathcal{F}) = \sum (-1)^{p+q} \dim H^p(Y, R^q f_*\mathcal{F})$$

Proof. This follows from 12.4.5 an induction. □

Exercises

1. Prove theorem 12.4.1.
2. Complete the proof of proposition 12.4.6. (See 10.4.2, for some hints.)
3. Show that equality holds in proposition 12.4.6 if and only of $\tau_{\leq \bullet}$ is strict.
4. Let $R^q = R^q f_*\mathcal{F}$. Define the map $d_2 : H^p(Y, R^q) \to H^{p+2}(Y, R^{q-1})$ as the composition of the connecting map associated to 12.4.5, and the map $H^*(\tau_{\leq q-1}) \to H^*(R^{q-1}[-q+1])$. This is the first step in the construction of Leray spectral sequence. Show that d_2 vanishes if $\tau_{\leq \bullet}$ is strict.

12.5. Estimate of first Betti number

Let us return to geometry, and compute the the first Betti number of an elliptic surface and more general examples. Let us set up some notation. Suppose that $f : X \to C$ is a map of a smooth projective variety onto a smooth projective curve. Assume that f has connected fibers. Let $S \subset C$ be the set of points for which the fibers are singular, and let U be its complement. The map $f^{-1}U \to U$ is a submersion and hence a fiber bundle. We have a monodromy representation of $\pi_1(U, y)$ on the the cohomomology of the fiber.

THEOREM 12.5.1. $b_1(C) \leq b_1(X) \leq b_1(C) + \dim H^1(X_y, \mathbb{Q})^{\pi_1(U)}$

COROLLARY 12.5.2. *If $\mathcal{E} \to C$ is an elliptic surface such that monodromy is nontrivial, then $b_1(\mathcal{E}) = b_1(C)$.*

The proof will be carried out in a series of steps. Let $j : U \to C$ denote the inclusion, and let $\mathcal{F} = R^1 f_* \mathbb{Q}$. There is a canonical morphism $\mathcal{F} \to j_* j^* \mathcal{F}$ induced by the restriction maps $\mathcal{F}(V) \to \mathcal{F}(V \cap U)$, as V ranges over open subsets of C.

An analytic hypersurface of a complex manifold is a subset which locally the zero set of a holomorphic function.

LEMMA 12.5.3. *Let $S \subset W$ be a compact analytic hypersurface of a complex manifold, then $H^1(W, \mathbb{Q}) \to H^1(W - S, \mathbb{Q})$ is injective.*

PROOF. By using Mayer-Vietoris and induction, we can reduce to the case where W is a ball. The result is obvious in this case. □

COROLLARY 12.5.4. *$\mathcal{F} \to j_* j^* \mathcal{F}$ is a monomorphism.*

PROOF. This follows for the injectivity of he restriction maps $H^1(V, \mathbb{Q}) \to H^1(V \cap U, \mathbb{Q})$. □

PROOF OF THEOREM. Proposition 12.4.6 gives

$$b_1(X) \leq \dim H^1(C, f_* \mathbb{Q}) + \dim H^0(C, R^1 f_* \mathbb{Q})$$

Since the fibers of f are connected, $f_* \mathbb{Q}$ is easily seen to be $\mathbb{Q}$. So the dimension of its first cohomology is just $b_1(C)$. The previous corollary implies that

$$H^0(C, R^1 f_* \mathbb{Q}) \to H^0(C, j_* j^* \mathcal{F}) = H^0(U, R^1 f_* \mathbb{Q}) = H^1(X_y, \mathbb{Q})^{\pi_1(U)}$$

is injective. This proves the upper inequallity

$$b_1(X) \leq b_1(C) + \dim H^1(X_y, \mathbb{Q})^{\pi_1(U)}$$

For the lower inequallity, note that a nonzero holomorphic 1-form on C will pullback to a nonzero form on X. Thus $h^{10}(C) \leq h^{10}(X)$. □

CHAPTER 13

The Hard Lefschetz Theorem

13.1. Hard Lefschetz and its consequences

Let X be an n dimensional compact Kähler manifold. Recall that L was defined by wedging (or more correctly cupping) with the Kähler class $[\omega]$. The space

$$P^i(X) = ker[L^{n-i+1} : H^i(X,\mathbb{C}) \to H^{2n-i+2}(X,\mathbb{C})]$$

is called the *primitive part* of cohomology.

THEOREM 13.1.1 (Hard Lefschetz). *For every i,*

$$L^i : H^{n-i}(X,\mathbb{C}) \to H^{n+i}(X,\mathbb{C})$$

is an isomorphism. For every i,

$$H^i(X,\mathbb{C}) = \bigoplus_{j=0}^{[i/2]} L^j P^{i-2j}(X)$$

We indicate the proof in the next section. As a simple corollary we find that the Betti numbers $b_{n-i} = b_{n+i}$. Of course, this is nothing new since this also follows Poincaré duality. However, it is easy to extract some less trivial "numerology".

COROLLARY 13.1.2. *The Betti numbers satisfy $b_{i-2} \leq b_i$ for $i \leq n/2$.*

PROOF. The theorem implies that the map $L : H^{i-2}(X) \to H^i(X)$ is injective. □

Suppose that $X = \mathbb{P}^n$ with the Fubini-Study metric 9.1.5. the Kähler class $\omega = c_1(\mathcal{O}(1))$. The class $c_1(\mathcal{O}(1))^i \neq 0$ is the fundamental class of a codimension i linear space (see sections 6.2 and 6.5), so it is nonzero. Since all the cohomology groups of $\mathbb{P}^n$ are either 0 or 1 dimensional, this implies the Hard Lefschetz theorem for $\mathbb{P}^n$. Things get much more interesting when $X \subset \mathbb{P}^n$ is a nonsingular subvariety with induced metric. By Poinaré duality and the previous remarks, we get a statement closer to what Lefschetz would have stated[1] Namely, that any element of $H_{n-i}(X,\mathbb{Q})$ is homologous to the intersection of a class in $H_{n+i}(X,\mathbb{Q})$ with a codimension i subspace.

The Hodge index for surfaces admits the following generalization called the Hodge-Riemman relations to an dimensional compact Kähler manifold X. Consider the pairing

$$H^i(X,\mathbb{C}) \times H^i(X,\mathbb{C}) \to \mathbb{C}$$

[1]Lefschetz stated a version of this theorem for varieties in his book [L] with an incomplete proof. The first correct proof is due to Hodge using harmonic forms. A subsequent arithmetic proof was given by Deligne [D5].

defined by

$$Q(\alpha, \beta) = (-1)^{i(i-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{n-i}$$

THEOREM 13.1.3. *$H^i(X) = \oplus H^{pq}(X)$ is an orthogonal decomposition with respect to Q. If $\alpha \in P^{p+q}(X) \cap H^{pq}(X)$ is nonzero,*

$$\sqrt{-1}^{p-q} Q(\alpha, \bar{\alpha}) > 0.$$

PROOF. See [**GH**, p.123]. □

Let us introduce the Weil operator $C : H^i(X) \to H^i(X)$ which acts on $H^{pq}(X)$ by multiplication by $\sqrt{-1}^{p-q}$.

COROLLARY 13.1.4. *The form $\tilde{Q}(\alpha, \beta) = Q(\alpha, C\bar{\beta})$ on $P^i(X)$ is positive definite Hermitean.*

Exercises

1. When X is compact Kähler, show that Q gives a nondegenerate skew symmetric pairing on $P^i(X)$ with i odd. Use this to another proof that $b_i(X)$ is even.
2. Determine which products of spheres $S^n \times S^m$ can admit Kähler structures.

13.2. More identities

Let X be as in the previous section. We defined the operators L, Λ acting forms $\mathcal{E}^\bullet(X)$ in section 9.1. We define a new operator H which acts by multiplication by $n - i$ on $\mathcal{E}^i(X)$. Then:

PROPOSITION 13.2.1. *The following hold:*

1. $[\Lambda, L] = H$
2. $[H, L] = -2L$
3. $[H, \Lambda] = 2\Lambda$

Furthermore these operators commute with Δ.

PROOF. See [**GH**, p 115, 121]. □

This plus the following theorem of linear algebra will prove the Hard Lefschetz theorem.

THEOREM 13.2.2. *Let V be a vector space with endomorphisms L, Λ, H satisfying the above idenities. Then*

1. *H is diagonalizable with integer eigenvalues.*
2. *For each $a \in \mathbb{Z}$ let V_a be the space of eigenvectors of H with eigenvalue a. Then L^i induces an isomomorphism between V_i and V_{-i}.*
3. *If $P = ker(\Lambda)$, then*

$$V = P \oplus LP \oplus L^2P \oplus \dots$$

4. *$\alpha \in P \cap V_i$ then $L^{i+1}\alpha = 0$.*

Consider Lie algebra $sl_2(\mathbb{C})$ of space of traceless 2×2 complex matrices. This is a Lie algebra with a basis given by

$$\lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\ell = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices satisfy

$$[\lambda, \ell] = h, \ [h, \lambda] = 2\lambda, \ [h, \ell] = -2\ell$$

So the hypothesis of the theorem is simply that the linear map determined by $\lambda \mapsto \Lambda$, $\ell \mapsto L$, $h \mapsto H$, preserves the bracket. Or equivalently that V is a representation of $sl_2(\mathbb{C})$. The theorem can then be deduced from the following two facts from representation theory of $sl_2(\mathbb{C})$ (which can be found in almost any book on Lie theory, e.g [**FH**]):

1. Every representation of $sl_2(\mathbb{C})$ is a direct sum of irreducible representations. Irreducibility means that the representation contains no proper nonzero subresentations, i.e. subspaces stable under the action of $sl_2(\mathbb{C})$.
2. The is a unique irreducible representation of dimension $N + 1$, for each $N \geq 0$, and it has has the following structure:

$$0 \xleftarrow[\lambda]{} V_N \cong \mathbb{C} \underset{\lambda \neq 0}{\overset{\ell \neq 0}{\rightleftarrows}} V_{N-2} \cong \mathbb{C} \underset{\lambda \neq 0}{\overset{\ell \neq 0}{\rightleftarrows}} \cdots \underset{\lambda \neq 0}{\overset{\ell \neq 0}{\rightleftarrows}} V_{-N} \cong \mathbb{C} \xrightarrow{\ell} 0$$

Exercises

1. Using the relations, check that $\lambda : V_a \to V_{a+2}$ and $\ell : V_a \to V_{a-2}$ for any representation V.
2. If V is a representation and $v \in ker(\lambda)$, show that $\{v, \ell v, \ell^2, \ldots\}$ spans a subrepresentation of V. In particular, this spans V if it is irreducible.

13.3. Lefschetz pencils

In this section, we explain Lefschetz's original approach to the hard Lefschetz theorem. This involves the comparison of the topology variety with a family of hyperplane sections. Modern references for this material are [**La**], [**SGA7**] and [**V**].

Let $X \subset \mathbb{P}^N$ be an n dimensional smooth projective variety, and let $Y = X \cap H$ be the intersection with general hyperplane. Bertini's theorem [**Har**] shows that Y is smooth. We define

$$V = ker[\tau_! : H^{n-1}(Y, \mathbb{Q}) \to H^{n+1}(X, \mathbb{Q})]$$

and

$$I = im[\tau^* : H^{n-1}(X, \mathbb{Q}) \to H^{n-1}(Y, \mathbb{Q})],$$

where $\tau : Y \to X$ is the inclusion. We will see toward the end of this section, that these are spaces of vanishing cycles and invariant cycles respectively.

Proposition 13.3.1. $H^{n-1}(Y) = I \oplus V$

PROOF. The composition

$$H^{n-1}(X) \to H^{n-1}(Y) \to H^{n+1}(X)$$

can be identified with the Lefschetz operator L. Therefore it is an isomorphism. It follows that τ^* is injective, and $\tau_!$ is surjective. Therefore, we have an exact sequence

$$0 \to V \to H^{n-1}(Y) \to H^{n+1}(X)$$

The map

$$\tau^* L^{-1} \tau_! : H^{n+1}(X) \to H^{n-1}(Y)$$

gives a splitting, and the proposition follows. □

When translated into cohomology, Lefschetz's original statement would more or less be previous proposition. However, this is quite far from his original formulation, which is quite geometric. Here we explain some of the ideas. Choose a smooth n dimensional projective variety $X \subset \mathbb{P}^N$. We can and will assume that X is nondegenerate which means that X does lie on a hyperplane. Let $\check{\mathbb{P}}^N$ be the dual projective space whose points correspond to hyperplanes of $\mathbb{P}^N$. The dual variety

$$\check{X} = \{H \in \check{\mathbb{P}}^N \mid H \text{ contains a tangent space of } X\}$$

parameterizes hyperplanes such that $H \cap X$ is singular.

PROPOSITION 13.3.2. *If $H \in \check{X}$ is a smooth point, then $H \cap X$ has exactly one singular point which is a node (i.e. the completed local ring is isomorphic to $\mathbb{C}[[x_1, \ldots x_n]]/(x_1^2 + \ldots x_n^2)$).*

PROOF. [**SGA7**, chap XVII]. □

A line $\{H_t\}_{t \in \mathbb{P}^1} \subset \check{\mathbb{P}}^N$ is called a pencil of hyperplanes. Any two hyperlanes will intersect in a common linear subspace call the base locus. A pencil $\{H_t\}$ is called Lefschetz if $H_t \cap X$ has at worst a single node for all $t \in \mathbb{P}^1$ and if the base locus $H_0 \cap H_\infty$ is transverse to X.

COROLLARY 13.3.3. *The set of Lefschetz pencils form a nonempty Zariski open set of the Grasmannian of lines in $\check{\mathbb{P}}^N$.*

PROOF. A general pencil will automatically satisfy the transversality condition. Furthermore, since the singular set $\dim \check{X}_{sing}$ has codimension at least two in $\check{\mathbb{P}}^N$, a general pencil will avoid it. □

Given a pencil, we form an incidence variety $\tilde{X} = \{(x,t) \in X \times \mathbb{P}^1 \mid x \in H_t\}$ The second projection gives a map onto $\mathbb{P}^1$ whose fibers are intersections $H \cap X$. There is a finite set S of $t \in \mathbb{P}^1$ with $p^{-1}t$ singular. Let $U = \mathbb{P}^1 - S$. Choose $t_0 \in U$, set $\tilde{X}_{t_0} = p^{-1}(t_0)$ and consider the diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi} & \tilde{X} \\ \uparrow{\scriptstyle \tau} & \nearrow{\scriptstyle \tilde{\tau}} & \downarrow{\scriptstyle p} \\ \tilde{X}_{t_0} & & \mathbb{P}^1 \end{array}$$

Choose small disks Δ_i around each $t_i \in S$, and connect these by paths γ_i to the base point t_0 (figure 1).

The space $p^{-1}(\gamma_i \cup \Delta_i)$ is homotopic to the singular fiber $\tilde{X}_{t_i} = p^{-1}(t_i)$.

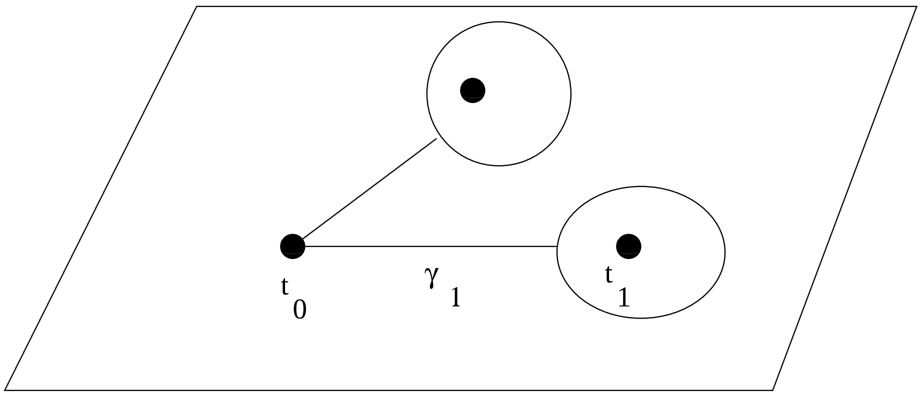


FIGURE 1. Loops

THEOREM 13.3.4. *Let $n = \dim X$. Then*

$$H_k(\tilde{X}_{t_0}, \mathbb{Q}) \to H_k(p^{-1}(\gamma_i \cup \Delta_i), \mathbb{Q})$$

is an isomorphism if $k \neq n-1$, and it is surjective with a one dimensional kernel if $k = n-1$.

PROOF. Since $p^{-1}(\gamma_i \cup \Delta_i)$ is homotopic to $p^{-1}(\Delta_i)$, we can assume that t_0 is a point on the boundary of Δ_i. Let $y_i \in \tilde{X}_i$ be the singular point. We can choose coordinates so that about y_i so that p is given by $z_1^2 + z_2^2 + \ldots z_n^2$. Let

$$B = \{(z_1, \ldots z_n) \mid |z_1|^2 + |z_2|^2 + \ldots |z_n|^2 < \epsilon^2,\ |z_1^2 + z_2^2 + \ldots z_n^2| < \rho\}$$

We assume, after shrinking Δ_i if necessary, that B surjects onto it. We identify t_0 with ρ. If $x = Re(z), y = Im(z)$, then we can identify

$$\tilde{X}_{t_0} \cap B = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid ||x||^2 + ||y||^2 < \epsilon^2,\ ||x||^2 - ||y||^2 = \rho,\ \langle x, y\rangle = 0\}$$

These inequalities imply that $||x|| \neq 0$ and $||y||^2 < (\epsilon^2 - \rho)/2$. Therefore $(x, y) \mapsto (x/||x||, 2y/(\epsilon^2 - \rho))$ gives homeomorphism

$$\tilde{X}_{t_0} \cap B \cong \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid ||x||^2 = 1,\ ||y||^2 < 1,\ \langle x, y\rangle = 0\}$$

The latter space deformation retracts on the sphere $S^{n-1} = \{(x, 0) \mid ||x|| = 1\}$. It follows that $ker[H_k(\tilde{X}_{t_0} \cap B) \to H_k(B)]$ is generated by the fundamental class of S^{n-1} when $k = n-1$ and an isomorphism otherwise. This is a local version of what we want. For the global version, we need to appeal to a refinement of Ehresmann's fibration theorem [La], which will imply that the bundle $(\tilde{X} - B, \partial(\tilde{X} - B)) \to \Delta_i$ is trivial. Thus the fibers $(\tilde{X}_w - B, \partial(\tilde{X}_w - B))$ with $w \in \Delta_i$, are all diffeomorphic. Thus, by excision, we have a commutative diagram

$$\begin{array}{ccccccc} H_{k+1}(\tilde{X}_{t_0}, \tilde{X}_{t_0} \cap B) & \to & H_k(\tilde{X}_{t_0} \cap B) & \to & H_k(\tilde{X}_{t_0}) & \to & H_k(\tilde{X}_{t_0}, \tilde{X}_{t_0} \cap B) \\ \| & & \downarrow & & \downarrow & & \| \\ H_{k+1}(Y \cap B) & \to & H_k(Y \cap B) & \to & H_k(Y) & \to & H_k(Y, Y \cap B) \end{array}$$

with isomorphisms at the ends. The theorem follows from this. $\square$

Let $\delta_i \in H_{n-1}(\tilde{X}_{t_0})$ be the homology class of S^{n-1} constructed above. It is called the vanishing cycle. Let $\mu_i : H_k(\tilde{X}_{t_0}) \to H_k(\tilde{X}_{t_0})$ denote the monodromy for the loop γ_i, followed by $\partial\Delta_i$ and then γ_i^{-1}. The problem of computing this can be localized to a neighbourhood of the singularity. Let B be as in the above proof. Given a cycle α, representing a class in $H_k(\tilde{X}_{t_0}, B\cap \tilde{X}_{t_0})$, $\mu_i(\alpha)$ is the effect of transporting α around the loop. Since the family $(Y-B, \partial(Y-B)) \to \Delta_i$ is trivial, μ_i acts trivially on the boundary. Thus the difference $\mu_i(\alpha)-\alpha$ is absolute cycle defining a class $var_i(\alpha)\in H_k(\tilde{X}_{t_0})$. This defines a map

$$var_i : H_k(\tilde{X}_{t_0}, B\cap \tilde{X}_{t_0}) \to H_k(\tilde{X}_{t_0})$$

such that the monodromy is given by

$$\mu_i = I + var_i\circ \iota$$

where $\iota : H_k(\tilde{X}_{t_0}) \to H_k(\tilde{X}_{t_0}, B\cap \tilde{X}_{t_0})$ is the natural map. It follows from this that

$$\mu_i(\alpha) = \begin{cases} \alpha + c(\alpha)\delta_i & \text{if } k = n-1 \\ \alpha & \text{otherwise}\end{cases}$$

for some constant $c(\alpha)$. The precise determination of the constant is given the Picard-Lefschetz formula. We state it in cohomology which is more convenient for our purposes. Identify the vanishing cycle with an element of cohomology via the Poincaré duality isomorphism $H_{n-1}(\tilde{X}_{t_0},\mathbb{Q}) \cong H^{n-1}(\tilde{X}_{t_0},\mathbb{Q})$. Then:

Theorem 13.3.5 (Picard-Lefschetz). *$\mu_i(\alpha) = \alpha + (-1)^{n(n+1)/2}\langle\alpha,\delta_i\rangle\delta_i$ where $\langle,\rangle$ denotes the cup product pairing on $H^{n-1}(\tilde{X}_{t_0},\mathbb{Q})$*

Proof. See [**La**] or [**Ar**, sect 2.4]. □

Let $Van \subset H^{n-1}(\tilde{X}_{t_0},\mathbb{Q})$ be the subspace spanned by all of the vanishing cycles δ_i. Let $Inv = H^{n-1}(\tilde{X}_{t_0})^{\pi_1(U)}$ be the space of of classes invariant under $\pi_1(U)$.

Corollary 13.3.6. *The orthogonal complement $Van^\perp$ coincides with Inv.*

Proof. $\mu_i(\alpha)=\alpha$ if and only if $\langle a,\delta_i\rangle = 0$. □

We now identify $Y=\tilde{X}_{t_0}$, and compare these spaces with the ones introduced at the beginning.

Proposition 13.3.7. *$V = Van$.*

Proof. The proof is adapted from [**La**]. Utilizing the Poincaré duality isomorphism $H_{n-1}(\tilde{X}_{t_0}) \cong H^{n-1}(\tilde{X}_{t_0})$, we can work in homology.

We divide $\mathbb{P}^1$ into upper and lower hemispheres U_+ and U_-. Without loss of generality, we can assume that all the singular fibers $\tilde{X}_{t_1},\ldots$ of $p:\tilde{X}\to\mathbb{P}^1$ are contained in $\tilde{X}_+ = p^{-1}(U_+)$, and t_0 lies on the boundary of U_+. Let Δ_i be disjoint disks around t_i contained in U_+. Let $r_i\in\partial\Delta_i$. Join these by paths γ_i. Let $Z=\cup p^{-1}\gamma_i$. This retracts to $\tilde{X}_{t_0}$. The diagram:

$$\begin{array}{ccccc} H_n(\tilde{X}_+,\tilde{X}_{t_0}) & \to & H_{n-1}(\tilde{X}_{t_0}) & & \\ \| & & \| & & \\ H_n(\coprod_{i=1}^m p^{-1}\Delta_i, Z) & \to & H_{n-1}(Z)\cong H_{n-1}(\tilde{X}_{t_0}) & \to & \bigoplus_{i=1}^m H_n(p^{-1}\Delta_i)\end{array}$$

shows that Van can be image of $H_n(\tilde{X}_+,\tilde{X}_{t_0})$. On the other hand the sequence

$$H_n(X,Y)\xrightarrow{b} H_{n-1}(Y)\to H_{n-1}(X)$$

and the fact that $\tau_!$ can be identified with map on the right shows that V is the image b. Now consider the diagram

$$\begin{array}{ccc} H_n(\tilde{X}_+, \tilde{X}_{t_0}) & \to & H_{n-1}(\tilde{X}_{t_0}) \\ f\downarrow & & \| \\ H_n(X,Y) & \to & H_{n-1}(Y) \end{array}$$

The map f is surjective [**La**, 3.6.5], and this implies that $V = Van$. □

PROPOSITION 13.3.8. $I = Inv$

PROOF. We prove this when n is odd. The image of $H^{n-1}(X,\mathbb{Q})$ lies in $H^{n-1}(\tilde{X}_{t_0})^{\pi_1(U)}$ since it factors through the edge homomorphism $H^{n-1}(\tilde{X},\mathbb{Q})$, section 12.4. Thus it suffices to prove equality of their dimensions. By corollary 13.3.6 and proposition 13.3.1

$$\dim Inv = \dim H^{n-1}(\tilde{X}_{t_0}) - \dim V = \dim I.$$

□

We finally state the following for later use.

THEOREM 13.3.9.
(a) *The restriction of the cup product pairing to I is nondegenerate.*
(b) *V is an irreducible $\pi_1(U)$-module, i.e. it has no $\pi_1(U)$-submodules other than 0 or V.*
(c) *$H^{n-1}(Y)$ is a semisimple $\pi_1(U)$-module.*

PROOF. The pairing on $I = imH^{n-1}(X)$ given by

$$\langle \tau^*\alpha, \tau^*\beta\rangle = \int_{\tilde{X}_{t_0}} \tau^*\alpha \cup \tau^*\beta = \pm Q(\alpha,\beta)$$

is nondegenerate by corollary 13.1.4. This proves (a).

We prove (b) modulo the basic fact that any two vanishing cycles are conjugate up to sign under the action of $\pi_1(U)$ [**La**, 7.3.5]. Suppose that $W \subset V$ is $\pi_1(U)$ submodule, let $w \in W$ be a nonzero element. By (a), $\langle w, \delta_i\rangle \neq 0$ for some i. Then

$$\mu_i(w) - w = \pm\langle w, \delta_i\rangle \delta_i \in W$$

implies that W contains δ_i, and thus all the vanishing cycles by the above fact. Therefore $W = V$.

By proposition 13.3.1, $H^{n-1}(Y) = I \oplus V$. The group $\pi_1(U)$ acts trivially on I by proposition 13.3.8 and irreducibly on V. This implies (c). □

13.4. Barth's theorem

Hartshorne [**Ha1**] gave a short elegent proof of Barth's theorem on cohomology of subvarieties of $\mathbb{P}^n$ as an application of Hard Lefschetz. We reproduce his argument here.

LEMMA 13.4.1. *Let $Y \subset X$ be oriented submanifold of a compact oriented C^∞ manifold. Then the following identities hold:*
(1) $\iota^*(\beta \cup \gamma) = \iota^*\beta \cup \iota^*\gamma$
(2) $\iota_!\iota^*\beta = [Y] \cup \beta$.
(3) $\iota^*\iota_!\alpha = i^*[Y] \cup \alpha$.

Proof. The first identity is simply a restatement of functoriality of cup product. The second is usually called the projection formula, and it follows from

$$\int_X \iota_! \iota^* \beta \cup \gamma = \int_Y \iota^* \beta \cup \iota^* \gamma = \int_X [Y] \cup \beta \cup \gamma$$

The same proof shows that the identity holds more generally for Y a compact submanifold of a noncompact manifold, since $i_!$ and $[Y]$ would take values in $H^*_c(X)$.

We can factor i as

$$Y \xrightarrow{j} T \xrightarrow{k} X$$

where T is tubular neighbourhood of Y. Let $\pi : T \to Y$ the projection. we have

$$\iota^* \iota_! \alpha = j^* k^* k_! j_! \alpha = j^* j_! \alpha$$

where $k_!$ is extension by 0. By the previous identity

$$j^* j_! \alpha = j^*(j_! j^*) \pi^* \alpha = j^*[Y] \cup j^* \pi^* \alpha = i^*[Y] \cup \alpha$$

□

Proposition 13.4.2. *If X is an n dimensional nonsingular complex projective variety, such that there are positive integers $m < n$ and $N \leq 2m - n$ for which*

$$1 = b_2 = b_4 = \ldots b_{2([N/2]+n-m)}$$

$$b_1 = b_3 = \ldots b_{1+2([(N-1)/2]+n-m)}$$

Then if $Y \subset X$ is a nonsingular m dimensional subvariety, the restriction map $H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i < N$.

Proof. The assumptions imply that $H^{2(n-m)}(X)$ is generated by L^{n-m}. Therefore $[Y] = d[H]^{n-m}$ with $d \neq 0$. Let $\iota : Y \to X$ denote the inclusion. Let L be the Lefschetz operator associated to H and $H|_Y$ (it will be clear from context, which is which). Consider the diagram

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{n-m}} & H^{i+2(n-m)}(X) \\ \downarrow \iota^* & \nearrow (1/d)\iota_! & \downarrow \iota^* \\ H^i(Y) & \xrightarrow{L^{n-m}} & H^{i+2(n-m)}(X) \end{array}$$

The diagram commutes thanks to the previous lemma.

Hard Lefschetz for X implies that $L^{n-m} : H^i(Y) \to H^{i+2(n-m)}(Y)$ is injective and hence an isomorphism by our assumptions. It follows that the restriction $\iota^* : H^i(X) \to H^i(Y)$ is injective. It's enough to prove equality of dimension. Hard Lefschetz for Y implies that $L^{n-m} : H^i(Y) \to H^{i+2(n-m)}(Y)$ is injective. Therefore the same is true of $\iota_!$. Therefore

$$b_i(X) \leq b_i(Y) \leq b_{i+2(n-m)}(X) = b_i(X)$$

□

Corollary 13.4.3 (Barth). *If $Y \subset \mathbb{P}^n$ is a nonsingular, complex projective variety, then $H^i(\mathbb{P}^n, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i < \dim Y$.*

When Y has codimension one, this is a special case of the Lefschetz hyperplane theorem.

13.5. Hodge conjecture

Let X be an n dimensional nonsingular complex projective variety. We define the space of codimension p Hodge cycles on X to be

$$H^{2p}_{hodge}(X,\mathbb{Q}) = H^{2p}(X,\mathbb{Q}) \cap H^{pp}(X)$$

and let $H^{2p}_{hodge}(X,\mathbb{Z})$ denote the preimage of $H^{pp}(X)$ in $H^{2p}(X,\mathbb{Z})$.

LEMMA 13.5.1. *Given a nonsingular subvariety $i : Y \to X$ of codimension p, the fundamental class $[Y] \in H^{2p}_{hodge}(X,\mathbb{Z})$*

PROOF. $[Y]$ corresponds under Poincaré duality to the functional $\alpha \mapsto \int_Y \alpha$. If α has type (a,b), then $\alpha|_Y = 0$ unless $a = b = p$. Therefore $[Y]$ has type (p,p) thanks to corollary 10.2.8. The class $[Y]$ is also integral, hence the lemma. □

Even if Y has singularities, a fundamental class can be defined with the above properties. Here we give a quick but nonelementary definition. Let us first observe.

LEMMA 13.5.2. *$\alpha \in H^{2p}_{hodge}(X,\mathbb{Z})$ if and only if $1 \mapsto \alpha$ defines a morphism of Hodge structures $\mathbb{Z}(0) \to H^{2p}(X,\mathbb{Z})(p)$. Consequently,*

$$H^{2p}_{hodge}(X) \cong Hom_{HS}(\mathbb{Z}(0), H^{2p}(X,\mathbb{Z})(p)).$$

By Hironaka's famous theorem [**Hrn**], there exists a smooth projective variety $\tilde{Y}$ with a birational map $p : \tilde{Y} \to Y$. Let $\tilde{i} : \tilde{Y} \to X$ denote the composition of p and the inclusion. By corollary 10.2.10, we have a morphism

$$\tilde{i}_! : H^0(\tilde{Y}) = \mathbb{Z} \to H^{2p}(X)(-p)$$

This defines a class $[Y] \in H^{2p}_{hodge}(X)$ which is easily seen to be independent of $\tilde{Y}$.

Let $H^{2p}_{alg}(X,\mathbb{Q}) \subseteq H^{2p}_{hodge}(X,\mathbb{Q})$ be the subspace spanned by fundamental classes of codimension p subvarieties. The (in)famous Hodge conjecture asserts:

CONJECTURE 13.5.3 (Hodge). *$H^{2p}_{alg}(X,\mathbb{Q})$ and $H^{p}_{hodge}(X,\mathbb{Q})$ coincide.*

Note that in the original formulation $\mathbb{Z}$ was used in place of $\mathbb{Q}$, but this is known to be false [**AH**]. For $p = 1$, the conjecture is true by the Lefschetz $(1,1)$ theorem 9.3.1. We prove it holds for $p = \dim, X - 1$.

PROPOSITION 13.5.4. *If the Hodge conjecture holds for X in degree $2p$ with $p < n = \dim X$, i.e. if $H^{2p}_{alg}(X,\mathbb{Q}) = H^{2p}_{hodge}(X,\mathbb{Q})$, then it holds in degree $2n-2p$.*

PROOF. Let L be the Lefshetz operator corresponding to a projective embedding $X \subset \mathbb{P}^N$. Then for any subvariety Y $L[Y] = [Y \cap H]$ where H is a hyperplane section chosen in general position. It follows thatL^{n-2p} takes $H^{2p}_{alg}(X)$ to $H^{2n-2p}_{alg}(X)$ and the map is injective. Thus

$$\dim H^{2p}_{hodge}(X) = \dim H^{2p}_{alg}(X) \le \dim H^{2n-2p}_{alg}(X) \le \dim H^{2n-2p}_{hodge}(X)$$

On the other hand L^{n-2p} induces an isomorphism of Hodge structures $H^{2p}(X,\mathbb{Q})(p-n) \cong H^{2n-2p}(X,\mathbb{Q})$, and therefore an isomorphism $H^{2p}_{hodge}(X,\mathbb{Q}) \cong H^{2n-2p}_{hodge}(X,\mathbb{Q})$. This forces equality of the above dimensions. □

COROLLARY 13.5.5. *The Hodge conjecture holds in degree $n-1$. In particular it holds for three dimensional varieties.*

13.6. Degeneration of Leray

We want to mention one last consequence of the Hard Lefschetz theorem due to Deligne [**D1**]. A projective morphism of smooth complex algebraic varieties is called smooth if the induced maps on (algbraic) tangent spaces are surjective. In particular, such maps are C^∞ fiber bundles.

THEOREM 13.6.1. *Let $f : X \to Y$ be smooth projective map of smooth complex algberaic varieties, then the inequalities in proposition 12.4.6 for $\mathcal{F} = \mathbb{Q}$ are equalities, i.e.*

$$\dim H^i(X, \mathbb{Q}) = \sum_{p+q=i} \dim H^p(Y, R^q f_* \mathbb{Q})$$

PROOF. Our proof, based on [**GH**, pp. 462-468], will be incomplete, but will give the basic idea. In the exercises of section 12.4, we constructed a map $d_2 : H^p(Y, R^q) \to H^{p+2}(Y, R^{q-1})$, where $R^q = R^q f_* \mathbb{Q}$, The vanishing of d_2 and the higher differentials (which we haven't constructed) is equivalent to the conclusion of the theorem. We do this for d_2 only.

Let n be the dimension of the fibers. We can apply the Lefschetz decomposition fiberwise:

$$L^i : R^{n-i} \cong R^{n+i}$$

$$R^i = \bigoplus_{j=0}^{[i/2]} L^j P^{i-2j}$$

where

$$P^i = ker[L^{n-i+1} : R^i \to R^{2n-i+2}]$$

This allows us to decompose

$$H^p(Y, R^q) \cong \bigoplus H^p(Y, P^{q-2j})$$

Thus it suffices to check vanishing of the restrictions of d_2 to these factors. Consider the diagram

$$\begin{array}{ccc} H^p(Y, P^{n-k}) & \xrightarrow{d_2} & H^{p+2}(Y, R^{n-k-1}) \\ \downarrow{\scriptstyle L^{k+1}} & & \downarrow{\scriptstyle L^{k+1}} \\ H^p(Y, P^{n+k+2}) & \xrightarrow{d_2} & H^{p+2}(Y, R^{n+k+1}) \end{array}$$

The first vertical arrow is zero by the definition of P, and the second vertical arrow is an isomorphism by Hard Lefschetz. Therefore the top d_2 vanishes. □

COROLLARY 13.6.2. *If the monodromy action of $\pi_1(Y, y)$ on the cohomology of the fiber X_y is trivial (e.g. if Y is simply connected) then*

$$\dim H^i(X, \mathbb{Q}) = \sum_{p+q=i} \dim H^p(Y) \otimes H^q(X_y)$$

Exercises

1. Let $\pi : \mathbb{C}^2 - \{0\} \to \mathbb{P}^1_{\mathbb{C}}$ be the usual projection. This is smooth and in fact a Zarsiki locally trivial $\mathbb{C}^*$-bundle. The restriction $S^3 \to \mathbb{P}^1_{\mathbb{C}}$ to the unit sphere is called the Hopf fibration. Show that the conclusion of corollary 13.6.2 fails for π and the Hopf fibration.

13.7. Higher Chern classes

A projective morphism $f : X \to Y$ of varieties is said to be Brauer-Severi if it is smooth and all the fibres are isomorphic to projective space. Fix such a morphism. Choose an embedding of X into projective space and let $h = c_1(\mathcal{O}_X(1))$.

LEMMA 13.7.1. *If $f : X \to Y$ is Brauer-Severi with n dimensional fibres then*

$$H^i(X,\mathbb{Q}) = \bigoplus_{j=0}^{n} H^{i-2j}(Y,\mathbb{Q})h^j$$

where h^j is the jth power under cup product.

PROOF. The spaces on the right are easily seen to be disjoint, so it is enough to check the dimensions of both sides coincide. The class h will be a restrict to a nonzero element of $H^2(X_y,\mathbb{Q}) = H^2(\mathbb{P}^n,\mathbb{Q}) \cong \mathbb{Q}$. It follows that $H^*(X_y)$ is generated by powers of h. Since h is invariant under monodromy, we can apply corollary 13.6.2 to get the equality of dimensions. □

The Brauer-Severi morphisms of interest to us arise as follows. Let E be a rank $n + 1$ algebraic vector bundle on Y in the original sense (as opposed to a locally free sheaf). Define

$$\mathbb{P}(E) = \{\ell \mid \ \ell \text{ a line in some } E_y\} = \bigcup \mathbb{P}(E_y)$$

This comes with a projection $\pi : \mathbb{P}(E) \to Y$, which sends ℓ to y. The fibre is precisely $\mathbb{P}(E_y)$. We can make this into a variety, with π a morphism, as follows. Let g_{ij} be a 1-cocycle for E with respect to a cover $\{U_i\}$. Then we construct $\mathbb{P}(E)$ by gluing $(y, [v]) \in U_i \times \mathbb{P}^n$ to $(y, [g_{ij}v]) \in U_j \times \mathbb{P}^n$. Consider the line bundle

$$L = \{(v, \ell) \mid v \in \ell,\ \ell \text{ a line in some } E_y\}$$

on $\mathbb{P}(E)$. We define $\mathcal{O}_{\mathbb{P}(E)}(1)$ as the sheaf of its sections. This need not be very ample, that is it need come from an embedding of $\mathbb{P}(E)$ into a projective space. However, $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*M$ will be very ample for some suitable M. Then we get

COROLLARY 13.7.2. *The above decomposition is valid for $X = \mathbb{P}(E)$ and $h = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$.*

It follows that h^{n+1} is a linear combination of $1, h, \dots h^n$ with coefficients in $H^*(Y)$. These coefficients are, by definition, the Chern classes of E. More precisely, we define the ith Chern class $c_i(E) \in H^{2i}(Y,\mathbb{Q})$ by

$$h^{n+1} + c_1(E) \cup h^n + \dots c_{n+1}(E) = 0 \tag{24}$$

This method of defining Chern classes is due to Grothendieck. We should point that our definition of $\mathbb{P}(E)$ is dual the one given in [**Har**], so our signs are different.

We want to end this one last application of Hard-Lefschetz. A vector bundle E is called *negative* if the $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample, and E is ample if E^* is negative.

THEOREM 13.7.3 (Bloch-Gieseker). *If E is a negative vector bundle of rank $n+1$ on a smooth projective variety X with $d = \dim X \le n+1$. Then $c_d(E) \neq 0$.*

PROOF. Let $h = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, and let

$$\eta = h^{d-1} + c_1(E)h^{d-1} + \dots c_{d-1}(E) \in H^{2d-2}(\mathbb{P}(E))$$

Since h is ample, the Hard Lefschetz theorem guarantees that

$$h^{n+2-d} \cup : H^{2d-2}(\mathbb{P}(E)) \to H^{2n+2}(\mathbb{P}(E))$$

is injective. We have

$$\eta \cup h^{n+2-d} = -c_d(E) \cup h^{n+1-d}$$

by (24) and the fact that $c_i(E) = 0$ for $i > d$. Therefore $c_d(E)$ cannot vanish.

□

Exercises

1. Verify that Chern classes are functorial, i.e. $c_i(f^*E) = f^*c_i(E)$. Conclude that if $E \cong \mathcal{O}_Y^{n+1}$, the Chern classes vanish.