Part 4

Mixed Betti and Hodge numbers

CHAPTER 17

Additive invariants

17.1. Basic examples

Let k be a field. Let Var_k denote the collection of all possibly reducible varieties over k. An *invariant* is a map $I : Var_k \to A$ to some set, such that $I(X) = I(Y)$ if and X and Y are isomorphic. We have many examples of invariants, such the dimension, the various cohomology groups and so on. We want to concentrate on invariants satisfying a property which makes them more computable.

Definition 17.1.1. *An invariant $I : Var_k \to A$ is called additive if A is an abelian group and if $I(X) = I(X - Z) + I(Z)$ whenever $Z \subset X$ is closed. We will call I multiplicative, if in addition, A is a ring and $I(X \times Y) = I(X)I(Y)$.*

By induction, we can see that

Lemma 17.1.2. *If $X = \cup X_i$ is a disjoint union of locally closed varieties, then $I(X) = \sum_i I(X_i)$ for any additive invariant.*

Example 17.1.3. *Since we can express $\mathbb{P}^m_k = \mathbb{A}^m_k \cup \mathbb{A}^{m-1}_k \cup \ldots$, we see that $I(\mathbb{P}^m) = I(\mathbb{A}^m) + I(\mathbb{A}^{m-1}) + \ldots$.*

Here are a few examples.

Example 17.1.4. *Let $k = \mathbb{F}_q$ be the finite field with q elements. Given a quasi-projective variety $X \subseteq \mathbb{P}^N$, let $X(\mathbb{F}_{q^n})$ to be the set of points of $\mathbb{P}^N_{\mathbb{F}_{q^n}}$ satisfying the equations (and inequalities) defining X. Let $N_n(X)$ be the number of points of $X(\mathbb{F}_{q^n})$. This is clearly additive and multiplicative.*

The additivity property can be used in computations. For example, from above, we see that

$$N_n(\mathbb{P}^m_{\mathbb{F}_q}) = q^{nm} + q^{n(m-1)} + \ldots q^n.$$

Define the Grothendieck group $K(Var_k)$ to be the quotient of the free abelian group generated by isomorphism classes $[X]$ in Var_k by the relations

$$[X] = [X - Z] + [Z]$$

whenever $Z \subset X$ is closed.

Example 17.1.5. *$X \mapsto [X]$ is additive and multiplicative. In fact, it is the universal example.*

17.2. Euler characteristics

Let $k = \mathbb{C}$. We defined compactly supported cohomology of manifolds with real coefficients using differential forms in section 4.3. We can define this with coefficients in any abelian group A for a (locally compact Hausdorff) topological space X by

$$H_c^i(X, A) = H^i(\bar{X}/\{*\}, A)$$

where $\bar{X} = X \cup \{*\}$ is the one point compactification. Note that $\bar{X}$ can be replaced by any (reasonable) compactification, say $\tilde{X}$. In particular, if X is compact we can take $\tilde{X} = X$ and thus $H_c^*(X) = H^*(X)$. From (13), we get a long exact sequence

$$\ldots H_c^i(X, A) \to H^i(\tilde{X}, A) \to H^i(\tilde{X} - X, A) \to H_c^{i+1}(X, A) \to \ldots \tag{32}$$

In De Rham cohomology, the first first map is given by extending a compactly supported form by zero.

Definition 17.2.1. *The Euler characteristic (with respect to compactly supported cohomology) is*

$$\chi(X) = \sum (-1)^i \dim H_c^i(X, \mathbb{R})$$

Lemma 17.2.2. *Let χ is an additive and multiplicative invariant.*

Proof. The additivity follows immediately from (32). The multiplicativity follows from the Künneth formula

$$H^i(X \times Y, \mathbb{R}) = \bigoplus_{j+l=i} H^j(X, \mathbb{R}) \otimes H^l(Y, \mathbb{R})$$

□

17.3. Mixed Euler characteristics

We want to emphasize that the definition and properties of the Euler characteristic are not specific to algebraic varieties, and hold for more general spaces. However, there are refinements available for varieties.

Theorem 17.3.1. *For each integer m, there is an additive invariant $\chi^{(m)}$: $Var_{\mathbb{C}} \to \mathbb{Z}$ and invariant satisfying*

1. $\chi(X) = \sum \chi^{(m)}(X)$.
2. $\chi^{(m)}(X) = (-1)^m b_m(X)$ *if X smooth and projective.*
3. $\chi^{(m)}(X \times Y) = \sum_{r+s=m} \chi^{(r)}(X)\chi^{(s)}(Y)$.

We will explain where this comes from in the next chapter. We can translate the above properties into additivity and multiplicativity for the so called virtual Poincaré polynomial

$$P_X(t) = \sum (-1)^m \chi^{(m)}(X) t^m,$$

and that

$$P_X(t) = \sum b_m(X) t^m$$

if X is smooth and projective. The existence of this function has the following surprising consequence.

Corollary 17.3.2 (Durfee). *If $X = \cup X_i$ and $Y = \cup Y_i$ are smooth projective varieties expressable as disjoint unions of locally closed subvarieties such that $X_i \cong Y_i$, then X and Y have the same Betti numbers.*

This is simply not true for more nonprojective varieties or more general spaces (exercises). This invariant can be refined even further.

THEOREM 17.3.3. *For each pair of integers (p, q), there exists additive invariants $\chi^{(p,q)} : Var_{\mathbb{C}} \to \mathbb{Z}$ such that*

1. $\chi^{(m)}(X) = \sum_{p+q=m} \chi^{(p,q)}(X)$
2. $\chi^{(p,q)}(X) = (-1)^{p+q} h^{pq}(X)$ *if X is smooth and projective (where $h^{pq}(X)$ is the usual Hodge number).*

This leads to a practical tool for computing Hodge and Betti numbers for projective varieties that can be decomposed into simpler pieces. As an example of this, let X be smooth projective variety of dimension n. Choose $x \in X$. We can define the blow up $Bl_x X$ by generalizing the construction in 11.1. This is a smooth projective variety with a morphism $\pi : Bl_x X \to X$ which is an isomorphism over $X - \{x\}$ and such that $\pi^{-1}(x) \cong \mathbb{P}^{n-1}$. Then the Hodge numbers of $Bl_x X$ are determined by

$$\begin{aligned} \chi^{(p,q)}(Bl_x X) &= \chi^{(p,q)}(\mathbb{P}^{n-1}) + \chi^{(p,q)}(X - \{x\}) \\ &= \chi^{(p,q)}(\mathbb{P}^{n-1}) + \chi^{(p,q)}(X) - \chi^{(p,q)}(\{x\}) \\ &= \begin{cases} \chi^{(p,q)}(X) + 1 & \text{if } p = q > 0 \\ \chi^{(p,q)}(X) & \text{otherwise.} \end{cases} \end{aligned}$$

COROLLARY 17.3.4 (Durfee). *If $X = \cup X_i$ and $Y = \cup Y_i$ are smooth projective varieties expressable as disjoint unions of locally closed subvarieties such that $X_i \cong Y_i$, then X and Y have the same Hodge numbers.*

Each of the ruled surfaces F_n, described in section 11.1, can be decomposed as a union of $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{A}^1$. Thus the Hodge numbers are the same as for $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and this is easy to compute.

Exercises

1. Let $X = \mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}$ and $Y = \mathbb{C}^2 - \{0\}$. Decompose both as a disjoint union of $\mathbb{C}^*$ and $\mathbb{C}^* \times \mathbb{C}$, but show that their Betti numbers differ.
2. Using similar arguments show that the compact manifolds $X = S^1 \times S^2$ and $Y = S^3$ can both be decomposed as a union of S^1 and $S^1 \times \mathbb{R}^2$.

CHAPTER 18

Mixed Hodge Structures

Deligne has extended Hodge theory to algebraic varieties which may be be singular or noncompact. Here we give a brief introduction to these ideas by concentrating on the purely numerical aspects with a view toward explaining the functions $\chi^{(p,q)}$ introduced earlier.

18.1. Resolution of singularities

Besides Hodge theory, the basic tool is Hironaka's theorem about the existence of resolution of singularities. First, we state some terminology. Given an irreducible variety Y, a resolution of singularities of X is a nonsingular variety X with a surjective birational (generically one to one) morphism $\pi : X \to Y$. If Y has several components, we take a resolution to mean a disjoint union of resolutions of the components of X together with the obvious map. An effective divisor $D \subset X$ of a nonsingular variety is a codimension one subscheme. This equivalent to giving an ideal sheaf $I_D = \mathcal{O}_X(-D)$ which is locally principal. We say D has *normal crossings* if for each $x \in X$, there are local analytic coordinates $z_1, \ldots z_n$ about x such the ideal $I_{D,x}$ is generated by a monomial in z_i's. We are mostly interested in reduced divisors, in which case the $I_{D,x}$ can be assumed to be generated by an expression of the form $z_1 z_2 \ldots z_m$. Although, not always included in the definition, we would also like to require that the irreducible components of D are nonsingular.

Theorem 18.1.1 (Hironaka). *If Y is a variety with a Zariski closed set $Z \subset Y$, then there exists a resolution of singularties $\pi : X \to Y$ such that $\pi^{-1}(Z)$ is a divisor with normal crossings. If Z contains the singular locus, then we can assume that $X - p^{-1}(Z) \to Y - Z$ is an isomorphism. If Y is (quasi-)projective then so is X.*

Corollary 18.1.2. *If U is a quasiprojective variety, then there exists a projective compactification X such that $X - U$ is a divisor with normal crossings.*

Proof. Let Y be a projective compactification. Apply the theorem with $Z = Y - U$. $\square$

18.2. Mixed Hodge structures

Recall 10.1 that a real Hodge structure of weight m is a bigraded vector space $H = H_{\mathbb{R}} \otimes \mathbb{C} = \oplus_{p+q=m} H^{pq}$ satisfying Hodge symmetry $\bar{H}^{pq} = H^{qp}$. Split mixed Hodge structures are sums of Hodge structures of different weights.

Definition 18.2.1. *A split real mixed Hodge structure consists of finite dimensional bigraded vector space $H = H_{\mathbb{R}} \otimes \mathbb{C} = \oplus H^{pq}$ satisfying Hodge symmetry. A split mixed Hodge structure consists of the above plus a choice of lattice $H_{\mathbb{Z}} \otimes \mathbb{R} = H_{\mathbb{R}}$.*

These form a category where morphisms are linear maps preserving the lattice and bigradings. This contains the categories of pure Hodge structures of each weight as full subcategories.

The notion of a mixed Hodge structure is a little more subtle. Since split mixed Hodge structures are sufficient for our purposes, we will be content to define the notion a real mixed Hodge structure to give the flavour of the subject. As a first step, given a split mixed Hodge structure, we can define the (decreasing) Hodge filtration by

$$F^p = \bigoplus_{a\geq b} H^{ab}$$

and the (increasing) weight filtration by

$$W_k = \bigoplus_{a+b\leq k} H^{ab}$$

It is not hard to see that these filtrations determine the bigrading.

Definition 18.2.2. *A real* mixed Hodge structure *consists of a finite dimension complex vector space H with a real structure $H_{\mathbb{R}}$, a decreasing filtration $F^\bullet$ and an increasing filtration $W_\bullet$ satisfying:*

1. *Each W_m is real: $\bar{W}_m = W_m$*
2. *The filtration induced by F on $Gr^W_m H = W_m/W_{m-1}$ satisfies the conditions of lemma 10.1.1.*

The second guarantees that $Gr^W_m H$ carries a pure Hodge structure of weight m. Thus we can define the (p,q)th Hodge number of H as the dimension of the (p,q)th part of $Gr^W_{p+q} H$. Mixed Hodge structures form a category in a natural way, such that the category of split mixed Hodge structures can be embedded in it as a full subcategory. Theses categories turn out to be abelian (which is by no means obvious). A sequence of mixed Hodge structures is exact precisely when it is exact as sequence of abelian groups. There is an exact functor going in the backwards direction:

$$H \mapsto Split(H) = \bigoplus_m Gr^W_m$$

The main invariant of interest to us in are the Hodge numbers, and they coincide for H and $Split(H)$. Thus we may as well pass to $Split(H)$. However, we should emphasize that H and $Split(H)$ are rarely isomorphic, thus we are loosing something by doing this.

18.3. Mixed Hodge numbers

The following is really an amalgam of various theorems in [**D2**].

Theorem 18.3.1 (Deligne). *To every complex algebraic variety X, there is a canonical mixed Hodge structure on $H^i_c(X,\mathbb{C})$ such that*

1. *If X is smooth and projective, then this coincides with the pure Hodge structure introduced in theorem 10.2.4.*
2. *If $U \subset X$ is a Zariski open subset of a projective variety, with $Z = X - U$, then the long exact sequence*

$$\dots H^i_c(U) \to H^i(X) \to H^i(Z) \to H^{i+1}_c(U)\dots$$

is compatible with mixed Hodge structures.

We introduce the mixed Hodge and Betti numbers

$$h^{i;(p,q)}(X) = \dim H^i(X)^{(p,q)}$$

by

$$b_i^{(m)} = \sum_{p+q=m} h^{i;(p,q)}(X)$$

To get some feeling for this, let us calculate the dimension of these invariants for smooth nonprojective curve U. We can find a smooth compactification X of genus g. Let $Z = X - U$, this is a finite set of say s points. The map $H_c^2(U) \to H^2(X)$ is an isomorphisms and $H_c^0(U) = 0$, thus we get

$$0 \to H^0(X)^{(0,0)} \to H^0(Z)^{(p,q)} \to H_c^1(U)^{(p,q)} \to H^1(X)^{(p,q)} \to 0$$

Which gives

$$h^{1;(0,0)}(U) = s - 1,\ h^{1;(1,0)}(U) = h^{1;(0,1)}(U) = g$$

We can now define

$$\chi^{(p,q)}(X) = \sum_i (-1)^i h^{i;(p,q)}(X)$$

$$\chi^{(m)}(X) = \sum_i b_i^{(m)}(X)$$

following [**DK**]. Theorems 17.3.1 and 17.3.3 will follow from this.

18.4. Complement of a smooth hypersurface

While a proof of theorem 18.3.1 will be out of reach, it is instructive to see where some of this structure comes from in some special cases. Let us consider an n dimensional smooth projective variety X with a nonsingular connected hypersurface $D \subset X$. Let $U = X - D$, and let $i : D \to X$ and $j : U \to X$ denote the inclusions. Let $\pi : T \to D$ be a tubular neigbourhood of D 4.4.2. Any differential form with compact support on U can be extended by 0 to X. Thus the sheaf of compactly supported forms $\mathcal{E}^\bullet_{cU}$ can be regarded as subsheaf of $\mathcal{E}^\bullet_X$. This lies in the kernel $\mathcal{K}^\bullet$ of the restriction map $\mathcal{E}^\bullet_X \to \mathcal{E}^\bullet_D$.

LEMMA 18.4.1. *$\mathcal{E}^\bullet_{cU}$ is quasi-isomorphic to $\mathcal{K}^\bullet$.*

PROOF. □

The long exact associated to

$$0 \to \mathcal{K}^\bullet \to \mathcal{E}^\bullet_X \to \mathcal{E}^\bullet_D \to 0$$

is just (32). Since we want the mixed Hodge structure to be compatible with it, this forces

$$Split(H_c^i(U)) = ker[H^i(X) \to H^i(D)] \oplus im[H^{i-1}(X) \to H^{i-1}(D)]$$

In particular, the mixed Hodge numbers can be expressed as

$$h^{pq;i}(U) = \begin{cases} \dim ker[H^q(\Omega^p_X) \to H^q(\Omega^p_D)] & \text{if } p+q = i \\ \dim im[H^q(\Omega^p_X) \to H^q(\Omega^p_D)] & \text{if } p+q = i-1 \\ 0 & \text{otherwise} \end{cases}$$

We want to replace these with holomorphic objects. We define $\Omega^p_X(*D) \subset j_*\mathcal{E}^p_U$ to be the sheaf of meromorphic p-forms which are holomorphic on U. $\Omega^p_X(\log D) \subset \Omega^p_X(*D)$ is the subsheaf of meromorphic forms α such that both α and $d\alpha$ have simple poles along D. If we choose local coordinates $z_1, \dots z_n$ so that D is defined by $z_1 = 0$. Then the sections of $\Omega^p_X(\log D)$, are locally spanned as an $\mathcal{O}_X$ module by

$$\{dz_{i_1} \wedge \dots dz_{i_p} \mid i_j > 1\} \cup \{\frac{dz_1 \wedge dz_{i_2} \wedge \dots dz_{i_p}}{z_1}\}$$

$\Omega^p_X(\log D)(-D)$ is the product of the ideal sheaf of D with the previous sheaf. Locally this is spanned by

$$\{z_1 dz_{i_1} \wedge \dots dz_{i_p} \mid i_j > 1\} \cup \{dz_1 \wedge dz_{i_2} \wedge \dots dz_{i_p}\}$$

These are exactly the forms vanishing along D. Thus

LEMMA 18.4.2. $\Omega^p_X(\log D)(-D) = ker[\Omega^p_X \to \Omega^p_D]$

Clearly $\Omega^\bullet_X(\log D)(-D)$ is a subcomplex of $\Omega^\bullet_X$. It is not difficult to see, using 10.5, that in the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^\bullet_X(\log D)(-D) & \to & \Omega^\bullet_X & \to & \Omega^\bullet_D & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{K}^\bullet & \to & \mathcal{E}^\bullet_X & \to & \mathcal{E}^\bullet_D & \to & \end{array}$$

the vertical maps are quasi-isomorphisms. Thus

$$\dots \to \mathbb{H}^i(\Omega^\bullet_X(\log D)(-D)) \to \mathbb{H}^i(\Omega^\bullet_X) \to \mathbb{H}^i(\Omega^\bullet_D) \to \dots$$

coincides with (32). We also have sequences

$$\dots \to H^q(\Omega^p_X(\log D)(-D)) \to H^q(\Omega^p_X) \to H^q(\Omega^p_D) \to \dots \tag{33}$$

PROPOSITION 18.4.3 (Deligne). *There is a noncanonical decomposition*

$$H^i_c(U, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega^p_X(\log D)(-D))$$

PROOF. It suffices to prove equality of dimensions. The sequence (33) implies that

$$\dim H^q(X, \Omega^p_X(\log D)(-D)) = h^{pq;p+q}(U) + h^{pq;p+q-1}(U)$$

and this does the trick. □

COROLLARY 18.4.4. *$H^i(X, \mathcal{O}_X(-D)) = 0$ if the $2n - i$th Betti number of U vanishes.*

PROOF. This follows from Poincaré duality: $H^i_c(U) \cong H^{2n-i}(U)^*$. □

This yields a special case of the Kodaira vanishing theorem. The method used is closely related to various "topological" proofs found by Esnault, Viehweg, Kollár and others (see [**EV**]). We say that

COROLLARY 18.4.5 (Kodaira). *If D is very ample (the intersection of X with a hyperplane under a projective embedding), then $H^i(X, \mathcal{O}_X(-D)) = 0$ for $i > 0$.*

PROOF. Since $X - D$ is affine and hence Stein, this follows corollary 10.5.4 □

18.5. Smooth varieties

Having come this far, we can outline the construction when U is a smooth quasi-projective variety. By resolution of singularities, U can be compactified by a smooth projective variety $X \supset U$, such that $D = X - U$ is a divisor with normal crossings. We can assume to be reduced. To simplify the discussion, let us assume that $D = D_1 \cup D_2$ has two smooth irreducible components D_i. Let $\mathcal{K}^\bullet$ be the kernel of the restriction map $\mathcal{E}_X^\bullet \to \mathcal{E}_{D_1}^\bullet \oplus \mathcal{E}_{D_2}^\bullet$. As before, $\mathcal{E}_{cU}$ is quasi-isomorphic to $\mathcal{K}^\bullet$. The inclusion can be extended to an exact sequence

$$0 \to \mathcal{K}^\bullet \to \mathcal{E}_X^\bullet \to \mathcal{E}_{D_1}^\bullet \oplus \mathcal{E}_{D_2}^\bullet \to \mathcal{E}_{D_1\cap D_2}^\bullet \to 0$$

which can be used to express

$$H_c^i(U) \cong \mathcal{H}^i(T^\bullet)$$

where

$$T^i = \mathcal{E}^i(X) \oplus \mathcal{E}^{i-1}(D_1) \oplus \mathcal{E}^{i-1}(D_2) \oplus \mathcal{E}^{i-2}(D_1 \cap D_2)$$

with differential

$$\partial(\alpha, \beta_1, \beta_2, \gamma) = (d\alpha, \alpha|_{D_1} + d\beta_1, \alpha|_{D_2} + d\beta_2, \beta_1|_{D_1\cap D_2} - \beta_2|_{D_1\cap D_2} + d\gamma)$$

At least if one is willing to disregard the integral stucture, the filtrations defining mixed Hodge structure can be read off from T. The Hodge filtration is given by

$$F^pH_c^i(U) = im[\mathcal{H}(F^p\mathcal{E}^i(X) \oplus F^p\mathcal{E}^{i-1}(D_1) \oplus F^p\mathcal{E}^{i-1}(D_2) \oplus F^p\mathcal{E}^{i-2}(D_1 \cap D_2))]$$

where

$$F^p\mathcal{E} = \bigoplus_{r\geq p} \mathcal{E}^{rs}$$

The weight filtration is given by

$$W_mH_c^i(U) = \begin{cases} H_c^i(U) & \text{if } m \geq i \\ im[\mathcal{H}^i(\mathcal{E}^{i-1}(D_1) \oplus \mathcal{E}^{i-1}(D_2) \oplus \mathcal{E}^{i-2}(D_1 \cap D_2)) & \text{if } m = i-1 \\ im[\mathcal{H}^i(\mathcal{E}^{i-2}(D_1 \cap D_2))] & \text{if } m = i-2 \\ 0 & \text{if } m < i-2 \end{cases}$$

Although, we have chosen to work with C^∞-forms, these filtrations can also be expressed holomorphically using a complex $\Omega_X^\bullet(\log D)(-D)$ construct as before.

CHAPTER 19

Varieties over finite fields

19.1. The Deligne-Weil bound

In chapter 17, we gave a number of examples of additive invariants, including counting functions for varieties over finite fields. It turns that the there is a deeper connection between invariants. In order to state these, we start with a complex quasiprojective algebraic variety X with a fixed embedding into $\mathbb{P}^N_{\mathbb{C}}$. If the defining equations (and inequalities) have integer coefficients, then we can reduce the equations modulo a prime p to get a quasiprojective "variety" X_p defined over the finite field $\mathbb{F}_p$ with p elements (recall that, up to isomorphism, there is exactly one such field). In general, by adjoining the coefficients of the defining equations of X to $\mathbb{Z}$, we obtain a finitely generated algebra $A \subset \mathbb{C}$. Choose a maximal ideal $Q \subset A$, then A/Q will be a finite field. Fix an isomorphism $A/Q \cong \mathbb{F}_q$. Now we can reduce the equations modulo Q to get a quasiprojective "variety" X_Q defined over $\mathbb{F}_q$. We can define $X_Q(\mathbb{F}_{q^n})$ to be the set of points of $\mathbb{P}^N_{\mathbb{F}_{q^n}}$ satisfying the equations (and inequalities) defining X. Let $N_n(X_Q) = N_n$ be the cardinality of this set. In more abstract terms, we have a scheme $\mathcal{X} \to Spec\, A$ called a model of X. The original variety X is the fiber product $\mathcal{X} \times_{Spec\, A} Spec\, \mathbb{C}$, and and X_Q is the scheme theoretic fiber over Q. Note that even if X is smooth, X_Q need not be; we say that X has "good reduction" at Q if this is the case. We have $X_Q(\mathbb{F}_{q^n}) = Hom_{schemes}(Spec\, \mathbb{F}_{q^n}, X_Q)$.

As the following example indicates, it is not usually possible to write down exact formulas for N_n. So we should seek qualitative information.

Example 19.1.1. *Consider the elliptic curve E defined by $zy^2 = x^3 - z^3$. This equation gives a model over the integers. Then $N_1(E_p) = p + 1$ if $p \equiv 2 \mod 3$ is an odd prime, but not in general as the following table shows:*

p	N_1	p	N_1	p	N_1	p	N_1
7	4	67	52	127	148	193	192
13	12	73	84	139	124	199	172
19	28	79	76	151	148	211	196
31	28	97	84	157	144	223	196
37	48	103	124	163	172	229	252
43	52	109	108	181	156	241	228

A quick inspection of the table suggests $N_1(E_p)$ stays fairly close to $1 + p$. In fact, we always have following estimate:

Theorem 19.1.2 (Weil). *If X is a smooth projective curve of genus g, and suppose that X has good reduction at Q. Then*

$$|N_n(X_Q) - (1 + q^n)| \leq 2g\sqrt{q}$$

This is very remarkable formula which says that topological and arithmetic properties of curves are related. Weil conjectured, and Deligne [**D4**] subsequently proved, that this phenomenon holds much more generally. To formulate it, let us say that an algebraic number $\lambda \in \bar{\mathbb{Q}}$ has uniform absolute value $x \in \mathbb{R}$ if $|\iota(\lambda)| = x$ for all embeddings $\iota : \bar{\mathbb{Q}} \to \mathbb{C}$.

THEOREM 19.1.3 (Deligne). *Let X be a smooth projective d dimensional variety and suppose that X has good reduction at Q. Then*

$$N_n(X_Q) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{b_i} \lambda_{ji}^n,$$

where λ_{ji} are algebraic integers with uniform absolute values $q^{i/2}$ and b_i coincides with the ith Betti number of X.

See [**Har**, appendix C] and especially [**K**] for a more involved discussion of this and the other Weil conjectures. It is worth noting that the classical method of Lefschetz pencils play an important role in the proof.

Deligne [**D3**], [**D5**] found a refinement for singular or open varieties.

THEOREM 19.1.4 (Deligne). *Let X be a d dimensional variety Then*

$$N_n(X_q) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{b_i} \lambda_{ji}^n$$

where λ_{ji} are algebraic integers with uniform absolute values in $\{0, q^{1/2}, q, \ldots q^{i/2}\}$. $b_i = \dim H_c^i(X, \mathbb{C})$. Furthermore the mixed Betti numbers

$$b_i^{(m)}(X) = \#\{j \mid |\lambda_{ij}| = q^{m/2}\}$$

19.2. ℓ-adic cohomology

The discussion in the previous section may have a seemed a bit like black magic. It may be worthwhile to explain a little more about the philosophy behind it. Let X be a smooth projective variety defined over $\mathbb{F}_q$, and $\bar{X}$ be the extension of X to the algebraic closure $\bar{\mathbb{F}}_q$. Fix a prime ℓ different from the characteristic of $\mathbb{F}_q$. If we choose an embedding $X \subset \mathbb{P}^N$, we have $F : \bar{X} \to \bar{X}$ be the Frobenius morphism which acts by raising the coordinates of the qth power (see [**Har**] for a more precise description). Then $N_n(X)$ is just the number of fixed points of F^n. Weil suggested that that these numbers could be computed by an appropriate generalization of Lefschetz's trace formula. This was realized by Grothendieck's ℓ-adic cohomology theory [**Mi**], where ℓ is fixed prime not dividing q. The theory assigns to X a collection of finite dimensional $\mathbb{Q}_\ell$-vector spaces $H_{et}^i(\bar{X}, \mathbb{Q}_l)$ (and $H_{et,c}^i(\bar{X}, \mathbb{Q}_l)$), which behaves very much singular cohomology of a complex variety (with compact support). In particular, these satisfy:

1. This cohomology theory satisfy analogues of the Künneth formulas, Poincaré duality, and the Lefschetz trace formula.
2. If X is obtained as in the previous section by reducing modulo Q, $\dim H_{et,c}^i(\bar{X}, \mathbb{Q}_l)$ coincides with the Betti number of the original complex variety.

The trace formula shows that

$$N_n(X) = \sum_i (-1)^i trace[F^{n*} : H_{et}^i(\bar{X}, \mathbb{Q}_l) \to H_{et}^i(\bar{X}, \mathbb{Q}_l)]$$

The λ_{ji} of theorems 19.1.3 and 19.1.4 are precisely the eigenvalues of F acting on these cohomology groups. Thus the real content of this theorem is the estimate on these eigenvalues. The W (weight) filtration in mixed Hodge theory admits an analogue in the ℓ-adic defined in terms of eigenspaces of F. The last part 19.1.4 amounts to the assertion that these spaces have the same dimension.

19.3. A transcendental analogue of Weil's conjecture

After this excursion into arithmetic, let us return to Hodge theory and prove an analogue of the Weil conjecture found by Serre [**S3**]. To set up the analogy let us replace $\bar{X}$ above by a smooth complex projective variety Y, and F by and endomorphism $f : Y \to Y$. As for q, if we consider, the effect of the Frobenius on $\mathbb{P}^N_{\mathbb{F}_q}$, the pullback of $\mathcal{O}(1)$ under this map is $\mathcal{O}(q)$. To complete the analogy, we require the existence of a very ample line bundle $\mathcal{O}_Y(1)$ on Y, so that $f^*\mathcal{O}_Y(1) \cong \mathcal{O}_Y(1)^{\otimes q}$. We can take $c_1(\mathcal{O}_Y(1))$ to be the Kähler class ω Then we have $f^*\omega = q\omega$.

Theorem 19.3.1 (Serre). *If $f : Y \to Y$ is holomomorphic endomorphism of a compact Kähler manifold with Kähler class ω, such that $f^*\omega = q\omega$ for some $q \in \mathbb{R}$. Then the eigenvalues λ of $f^* : H^i(Y, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ are algebraic integers with uniform absolute value $q^{i/2}$*

Proof. The theorem holds for $H^{2n}(Y)$ since ω^n generates it. By hypothesis, f^* preserves the Lefschetz decomposition (theorem 13.1.1), thus we can replace $H^i(Y)$ by primitive cohomology $P^i(Y)$. Recall from corollary 13.1.4,

$$\tilde{Q}(\alpha, \beta) = Q(\alpha, C\bar{\beta})$$

is positive definite Hermitean form $P^i(Y)$, where

$$Q(\alpha, \beta) = (-1)^{i(i-1)/2} \int \alpha \wedge \beta \wedge \omega^{n-i}.$$

Consider the endomorphism $F = q^{-i/2} f^*$ of $P^i(Y)$. We have

$$Q(F(\alpha), F(\beta)) = (-1)^{i(i-1)/2} q^{-n} \int f^*(\alpha \wedge \beta \wedge \omega^{n-i}) = Q(\alpha, \beta)$$

Moreover, since f^* is a morphism of Hodge structures, it preserves the Weil operator C. Therefore F is unitary with respect to $\tilde{Q}$, so its eigenvalues have norm 1. This gives the desired estimate on absolute values of the eigenvalues of f^*. Since f^* can be represented by an integer matrix, the set of it eigenvalues is a Galois invariant set of algebraic integers, so these have uniform absolute value $q^{i/2}$. □

Grothendieck suggested that one should be able to carry out a similar proof for the Weil conjectures. However making this work would require the full strength of his *standard conjectures* [**GSt, Kl**] which are, at present, very much open.

Bibliography

[Ar] V. Arnold, S. Gusein-Zade, A. Varchenko, *Singularities of differentiable maps II*, Birkhäuser (1988)

[AH] M. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology 1 (1962)

[AM] M. Atiyah, I. Macdonald, *Commutative algebra*, Adison-Wesley (1969)

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag (1984)

[BG] S. Bloch, D. Gieseker, *The positivity of Chern classes of an ample vector bundle*, Invent. math (1971)

[Ch] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press (1984)

[DK] V. Danilov, A. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers* Math. USSR Izv. (1987)

[D1] P. Deligne, *Théorème de Lefschetz et critéres de dégénérescence de suites spectrales* Publ. IHES 35 (1968)

[D2] P. Deligne, *Theorie de Hodge II, III*, Publ. IHES, (1972, 1974)

[D3] P. Deligne, *Poids dans la cohomologie des variétés algébriques.*, Proc. ICM (1974)

[D4] P. Deligne, *Conjecture de Weil I,*, Publ. IHES, (1974)

[D5] P. Deligne, *Conjecture de Weil II*, Publ. IHES, (1980)

[DGMS] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. 29 (1975)

[SGA7] P. Deligne, N. Katz, *SGA 7, II*, Lect. Notes 340, Springer-Verlag (1973)

[E] D. Eisenbud, *Commutative algebra* Springer-Verlag (1994)

[EH] D. Eisenbud, J. Harris,*Geometry of schemes*, Springer-Verlag (2000)

[EV] H Esnault, E. Viehweg, *Lectures on vanishing theorems*, Birkhauser (1993)

[M2] D. Eisenbud, D. Grayson, M. Stillman, B. Sturmfels, *Computations in algebraic geometry with Macaulay 2* Springer-Verlag (2001)

[F1] W. Fulton, *Intersection theory*, Springer-Verlag (1984)

[F2] W. Fulton, *Toric varieties* Princeton U. Press (1993)

[FH] W. Fulton, J. Harris, *Representation theory* Springer-Verlag (1991)

[GM] S. Gelfand, Y. Manin, *Methods of homological algebra*, Springer-Verlag (1996)

[G] R. Godement, *Toplogique algébriques et théorie de faiseaux*, Hermann (1964)

[GR] H. Grauert, R. Remmert, *Coherent analytic sheaves*, Springer-Verlang (1984)

[GPR] H. Grauert, T. Peternell, R. Remmert (Ed.) *Several complex variables VII*, Springer-Verlag (1991)

[GH] P. Griffiths, J. Harris, *Principles of algebraic geometr*y, John-Wiley (1978)

[GS] P. Griffiths, W. Schmid, *Recent developements in Hodge theory*, Proc. Bombay colloq. on Discrete subgroups of Lie groups, Tata Inst. (1975)

[EGA] A. Grothendieck, J. Dieudonné, *Élement de Géométrie Algébrique I, II, III, IV*, Publ IHES (1961-1967)

[Gr] A. Grothendieck, *Sur quelque point d'algebra homologique*, Tohoku Math. J. (1957)

[GSt] A. Grothendieck, *Standard conjectures on algebraic cycles*, Alg. Geom., Bombay 1968, Tata Inst. (1969)

[Hrs] J. Harris, *Algebraic geometry. A first course*. Springer-Verlag, (1992)

[HM] J. Harris, I. Morrison, *Moduli of curves*, Springer-Verlag (1998)

[Ha1] R. Hartshorne, *Varieties of small codimension in projective space*, Bull. AMS (1974)

[Har] R. Hartshorne, *Algebraic Geometry,* Springer-Verlag (1977)

[Hat] A. Hatcher, *Algebraic topology* Cambridge U. P (2002)

[Hrn] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic* 0, Ann. Math (1964)

[Hrz] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag (1978)

[Ho1] W. Hodge, *The theory and applications of harmonic integrals.* Cambridge (1941)

[Ho2] W. Hodge, *The topological invariants of algebraic varieties* , Proc. ICM (1950)

[Hrm] L. Hörmander, *An introduction to several complex variables* North Holland (1973)

[I] B. Iverson, *Cohomology of Sheaves*, Springer-Verlag (1986)

[K] N. Katz, *An overview of Deligne's proof of the Riemann hypothesis for varieties over finite fields.* Proc. Sympos. Pure Math., Vol. XXVIII, AMS (1976)

[Ke] J. Kelly, *General topology*, Springer-Verlag

[Kl] S. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Esposes sur la cohomologies les schemas, North Holland (1968)

[La] K. Lamotke, *The topology of complex projective varieties after S. Lefschetz.* Topology (1981)

[Ln] S. Lang, *Algebra, 3rd ed*, Springer-Verlag (2002)

[L] S. Lefschetz, *L'Analysis Situs et Géometrie Algébrique*, Gauthier-Villars (1923)

[Lw] J. Lewis, *A survey of the Hodge conjecture*, CRM monogarph, AMS (1999)

[MR] A. Milgram, P. Rosenbloom, *Harmonic forms and Heat conductionI*, Proc. Nat. Acad. Sci. (1951)

[Mi] J. Milne, *Etale cohomology*, Princeton U. P (1980)

[Ms] E. Moise, *Geometric topology in dimensions* 2 *and* 3. Springer-Verlag (1977)

[Mo] B. Moishezon, *On n-dimensional compact varieties with n algebraically independent meromorphic functions*, Izv. Akad Nauk SSSR (1966)

[MK] J. Morrow, K. Kodaira, *Complex Manifolds*, Holt, Rinehart, Winston (1971)

[M] D. Mumford, *Red Book of Varieties and Schemes,* Springer-Verlag (1988)

[P] V. Patodi, *Curvature and the eigenforms of the Laplace operator*. J. Differential Geometry 5 (1971)

[S1] J. P. Serre, *Faiseaux algebriques coherents*, Ann. Math. (1955)

[S2] J. P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (1956)

[S3] J. P. Serre, *Analogue Kählerian de Certaines Conjecture de Weil*, Ann. Math (1960)

[S4] J. P. Serre, *A course in arithmetic*, Springer-Verlag (1973)

[Si] J. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Springer-Verlag (1994)

[Sh] I. Shafarevich, *Basic Algebraic Geometry* Springer-Verlag (1977)

[Sp] E. Spanier, *Algebraic topology,* McGraw-Hill (1966)

[Spv] M. Spivak, *A comprehensive introduction to differential geometry I*, Publish or Perish (1970)

[V] C. Voisin, *Hodge theory and complex algebraic geometry I,II*, Cambridge (2002,2003)

[Wa] F. Warner, *Foundations of differentiable manifolds and Lie groups.* Springer-Verlag (1983)

[Wl] C. Weibel, *An introduction to homological algebra* Cambridge (1994)

[W] R. Wells, *Differential analysis on complex manifolds*, Springer-Verlag (1979)