COTLAR MARTINGALE TRANSFORMS AND RELATED SINGULAR INTEGRALS

RODRIGO BAÑUELOS

ABSTRACT. The "magical" identity discovered by M. Cotlar in 1955 for the Hilbert transform is established here in the setting of martingale transforms and, in particular, for conformal martingales. This, together with the probabilistic representation of the Riesz transforms, shows that, at the level of martingale transforms and in odd dimensions, they exhibit the same analytic-type structure as the Hilbert transform on the real line. Consequently, Cotlar's proof of the sharp L^p inequality for powers of 2 applies. The significance of the martingale Cotlar identity, whose proof is entirely elementary, does not lie in providing an alternative proof of this well-known and relatively simple estimate, but rather in the structural viewpoint it reveals. This structure is explored further.

Independent of Cotlar's identity, asymptotic bounds for the L^p norm of the vector of Riesz transforms are investigated. It is shown that, in the limit as $p \to \infty$, this norm coincides asymptotically with that of the Hilbert transform on the real line.

The study of the Cotlar identity in the martingale setting is motivated by the desire to gain new insight into two longstanding open problems: T. Iwaniec's 1983 conjecture on the norm of the Beurling-Ahlfors operator and the problem of determining the sharp constant in E. M. Stein's 1984 inequality for the vector of Riesz transforms. Related problems are also discussed.

The paper contains both a survey of known results and new contributions. An effort has been made to keep the exposition as self-contained as possible and to present the material in an accessible, largely expository style.

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Date: April 23, 2026.

1. INTRODUCTION

The purpose of this work is to present a Cotlar identity for martingales and to describe its relationship to the classical Riesz transforms on $\mathbb{R}^d$. Although, as we shall see, a Cotlar identity does not hold for the Riesz transforms, their representation à la Gundy-Varopoulos [56] shows that, at the level of martingale transforms and in odd dimensions, a conformal Cotlar-type structure is nevertheless present. More precisely, when d is odd, the Riesz transforms arise as conditional expectations of martingale transforms associated with certain $(d+1)\times(d+1)$ matrices which we call *Cotlar matrices* and for which the Cotlar martingale identity holds.

These matrices naturally extend to $\mathbb{C}^n$, with $n = d+1$, the structural features underlying the Hilbert transform and its connection with analytic functions in the complex plane; see Remark 4.14, Example 4.20, and equation (4.32). As a consequence of this connection, we obtain the known sharp L^p bounds for the Riesz transforms when $p = 2^n$, $n \in \mathbb{N}$, exactly as in Cotlar's classical result for the Hilbert transform and without appealing to the method of rotations, the Pichorides inequality [80], martingale inequalities from [5] or Bellman function techniques [91]. These ideas also apply to other geometric settings where no method of rotations is available, including the discrete operators studied in [13, 14, 21, 22] which use the inequalities form [5].

The now extensive literature, see for example [9, 10, 13, 18, 27, 27, 40] and the many reference contained therein, on applications of martingale techniques to Riesz transform and related singular integrals motivates the following questions. Is there a martingale version of Cotlar's identity? Could such a martingale identity, together with the probabilistic representation of the Riesz transforms, lead to applications in higher dimensions? In particular, might a martingale Cotlar formula shed new light on the over 40-year old unsolved problems:

(1) the sharp L^p bounds for the vector of Riesz transforms (E. M. Stein [85]), and
(2) the conjecture of T. Iwaniec [60] concerning the L^p norm of the Beurling-Ahlfors operator on $\mathbb{C}$,

even for special values of p such as, for example, powers of 2?

Both problems, and especially (2), have been studied extensively over the past four decades, with martingale inequalities playing a central role in many of the key developments. From the martingale point of view, which fails to yield even the simplest case of $p = 2$ that follows trivially from Fourier transform, the level of abstraction required to make further progress on these problems seems remarkably similar as discussed below.

1.1. **Notation.** Throughout we will mainly work with smooth real-valued functions of compact support. That is, we assume that $f : \mathbb{R}^d \to \mathbb{R}$ and $f \in C_0^\infty(\mathbb{R}^d)$. Once the L^p inequalities of interest are proved for this class of functions the usual density arguments give the results for all functions in L^p. There will also be occasions when $f : \mathbb{R}^d \to \mathbb{C}$ and this will be explicitly stated.

The standard notion $\|f\|_{L^p(\mathbb{R}^d)}$ is used for the p-norm of functions in $L^p(\mathbb{R}^d)$. By abuse of notation, $\|T\|_{L^p(\mathbb{R}^d)}$ will denote the norm of an operator $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$. A notable exception to this will be the norm of the Hilbert transform H which will always be denoted by $\|H\|_p$. For functions $f : \mathbb{R}^d \to \mathbb{C}$ we will write $\|f\|_{L^p(\mathbb{R}^d;\mathbb{C})}$ and similarly for operator norms.

For a random variable X defined on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ we will use the notation $\|X\|_p$ for its p-norm.

In what follows $1 < p < \infty$ and $p^* = \max\{p, q\}$, where q is the conjugate exponent of p. Define

$$p^* - 1 = \begin{cases} \frac{1}{p-1}, & 1 < p \leq 2, \\ p - 1, & 2 \leq p < \infty. \end{cases} \tag{1.1}$$

The constant $p^* - 1$, which is the sharp constant for martingale transforms [32], is often called the Burkholder constant. Similarly the constant

$$\|H\|_p = \cot\left(\frac{\pi}{2p^*}\right) = \begin{cases} \tan\left(\frac{\pi}{2p}\right), & 1 < p \leq 2 \\ \cot\left(\frac{\pi}{2p}\right), & 2 \leq p < \infty, \end{cases} \tag{1.2}$$

which is the p-norm of the Hilbert transform is often called the Pichorides constant [80] (independently found by B. Cole [48]).

Our normalization for the Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi x \cdot \xi} dx, \quad f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{-2\pi x \cdot \xi} d\xi.$$

2. Cotlar's "magical" identity

The Hilbert transform H is the most basic singular integral defined on $\mathbb{R}$ by

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-y)}{y} dy.$$

The following well-known and "magical" identity was proved by M. Cotlar in [37, pg 159]:

$$|Hf|^2 = 2H(fHf) + |f|^2. \tag{2.1}$$

There are other variants of this identity in the literature including those for discrete and free Hilbert transforms. See for example [13, 14, 53, 70] and references given there.

Cotlar's proof of (2.1) is simple using either analytic function arising from the harmonic extensions of f and Hf to the upper half-space, or by taking Fourier transforms of both sides and using the fact that

$$\widehat{Hf}(\xi) = i\,\mathrm{sign}(\xi)\widehat{f}(\xi) \tag{2.2}$$

For completeness, and since this argument will be adapted to the setting of conformal martingales, we outline the proof based on analytic functions. Let $u_f(x, y)$ be the harmonic extension of f to the upper half-space–its convolution with the Poisson kernel $P_y(x)$. Set $z = x + iy$ and consider the analytic function $F(z) = u_f(z) + iv_f(z)$, with $v_f(z)$ the conjugate harmonic function of u_f. Since F is analytic so is $F^2 = u_f^2 - v_f^2 + 2iu_f v_f$. Set

$$U = u_f^2 - u_{Hf}^2, \quad \text{and} \quad V = 2u_f u_{Hf}.$$

Recall that $v_f(z) = u_{Hf}(z)$, or equivalently that $v_{Hf} = u_{-f}$, using the fact that $H^2 = -I$. Taking boundary values we obtain $|f|^2 - |Hf|^2 = -2H(f\,Hf)$ and rearranging gives (2.1).

An application of this identity gives a simple proof of the L^p boundedness of H with the optimal constant for $p = 2^k$, $k \in \mathbb{N}$. Indeed, applying the Minkowski and Cauchy-Schwarz inequalities, it follows from (2.1) that

$$\begin{aligned}\|Hf\|_{2p}^2 = \|(Hf)^2\|_p &\leq \|f^2\|_p + 2\|H\,(f \cdot Hf)\,\|_p \\ &\leq \|f\|_{2p}^2 + 2\|H\|_p\,\|f \cdot Hf\|_p \\ &\leq \|f\|_{2p}^2 + 2\|H\|_p\,\|f\|_{2p}\,\|Hf\|_{2p} \\ &\leq \|f\|_{2p}^2 + 2\|H\|_p\,\|H\|_{2p}\,\|f\|_{2p}^2.\end{aligned}$$

This gives the inequality

$$\|H\|_{2p} \leq \|H\|_p + \sqrt{1 + \|H\|_p^2}.$$

Note that the last inequality is in fact valid for any $1 < p < \infty$. From $\|H\|_2 = 1$, induction on n and the trigonometric identity

$$\cot\left(\frac{\alpha}{2}\right) = \cot(\alpha) + \sqrt{1 + \cot^2(\alpha)},$$

it follows that

$$\|H\|_p \leq \cot\left(\frac{\pi}{2p}\right), \quad p = 2^k, \ \ k = 1, 2, \ldots. \tag{2.3}$$

By duality, a similar bound holds for $p = \frac{2^k}{2^k-1}$ with p in $\cot(\cdot)$ replaced by its conjugate exponent. That is, this argument gives Pichorides bound for powers of 2.

For some history of the use of the trigonometric identity in the context of Hilbert transforms in other settings, see [14,51,52] and references therein.

Except for the best constant that arises from the (clever) connection to the trigonometric identity the above argument and the Marcinkiewicz interpolation theorem is what Cotlar used to give a proof of the boundedness of the Hilbert transform on L^p, for $1 < p < \infty$.

Remark 2.1. *It may be of interest to note that in his original proof of the boundedness of the Hilbert transform (conjugate function), Riesz* [82] *first established the result for $p = 2m$ by using the fact that if f is analytic so is f^p, and then applying the Marcinkiewicz interpolation theorem; see* [54, p. 213].

For $j = 1, 2, \ldots, d$ and $f \in L^p(\mathbb{R}^d)$, the classical Riesz transforms on $\mathbb{R}^d$ are the singular integrals defined by

$$R_j f(x) = c_d \ p.v. \int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} f(x-y) dy, \qquad c_d = \pi^{-\frac{d+1}{2}} \Gamma(\tfrac{d+1}{2}), \tag{2.4}$$

with their Fourier transform given by

$$\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|} \widehat{f}(\xi). \tag{2.5}$$

When $d = 1$ both formulations agree with the Hilbert transform. Since the Riesz transforms or $\mathbb{R}^d$, $d > 1$, are Fourier multipliers extensions of the Hilbert transform, it follows from de Leeuw's extension theorem, [39], [54, Theorem 2.5.15], that $\|H\|_p \le \|R_j\|_{L^p(\mathbb{R}^d)}$. On the other hand, the Calderón-Zygmund method of rotations gives that $\|R_j\|_{L^p(\mathbb{R}^d)} \le \|H\|_p$. Thus,

$$\|R_j\|_{L^p(\mathbb{R}^d)} = \|H\|_p = \cot\left(\frac{\pi}{2p^*}\right), \quad 1 < p < \infty. \tag{2.6}$$

The equality (2.6) was first verified in [61] with this argument. This fact (i.e., (2.6)) will be use several times below.

In [5], sharp inequalities for orthogonal martingales are proved from which the upper bound inequality (2.6) follows. A fair questions that one may ask here is: given the simplicity of the above proof based on the method of rotation, why the need for martingale inequalities? One advantage of the martingale approach is their applications to Riesz transforms on different geometric settings beyond $\mathbb{R}^d$, as the literature already cited shows.

It follows from Cotlar's Fourier transform proof of (2.1), [37, p. 155], that if T_m is Fourier multiplier with the function $m \in L^\infty(\mathbb{R}^d)$ then the identity for T_m replacing H is equivalent to

$$(m(\xi+\gamma) - m(\xi))\,(m(-\xi) - m(\gamma)) = 0, \ \text{ a.e. } \xi, \gamma \in \mathbb{R}^d. \tag{2.7}$$

This imposes rigid conditions on the multiplier that are rarely satisfied outside of simple variations of the Hilbert transform. With $m_j(\xi) = \frac{i\xi_j}{|\xi|}$, it is straightforward to verify that Cotlar's identity fails for the Riesz transforms. For example, take $j = 1$, $\xi = e_1$, $\eta = e_2$ (the standard unit vectors) and use continuity at (ξ, η) to find a neighborhood of this point where the identity does not hold. Similarly, there is no Cotlar identity for the Beurling-Ahlfors operator or various other classical Fourier multiplier operators.

Despite the failure of Cotlar's identity for Riesz transforms, it is shown here that a version of the identity does hold in odd dimensions for a special subclass of orthogonal martingales transforms. Moreover, this identity, together with their probabilistic representation, can be used to prove the sharp L^p-bound for the Riesz transforms when $p = 2^n$ without appealing to the method or rotations, sharp martingale inequalities or other Burkholder-Bellman function techniques.

The proof of the martingale Cotlar identity is entirely elementary, relying only on the basic form of Itô's formula applied to the product of two martingales. Nevertheless, applying the martingale identity to the Riesz transform requires viewing the martingale transform in a different light than what has typically been used in previous applications. In particular, the martingale Cotlar identity developed here isolates the role of conformality at the martingale level for Riesz transforms in odd dimensions.

Since the application treated here, the sharp L^p bound for $p = 2^k$ for the Riesz transforms, is a special case of (2.6) which admits a simple analytic proof via the method of rotations and also follows from the martingale inequalities in [5], the question raised above on the need for martingale for the upper bound in (2.6) becomes even more compelling. The value of the martingale Cotlar identity therefore does not lie in providing an alternative proof for the special case of this estimate, but rather in the structural viewpoint it reveals. Such identities expose hidden algebraic relationships at the martingale level and may extend to situations where analytic methods are unavailable. This perspective may be particularly relevant in settings such as in [13], where Cotlar-type identities and inequalities play a role in identifying the norm of the discrete Riesz–Titchmarsh Hilbert transform for $p = 2k$, $k \in \mathbb{N}$.

3. COTLAR'S IDENTITY FOR MARTINGALES

Let us first recall two inequalities for martingales with continuous paths. Fix $d \geq 2$ and let $B_t = (B_t^1, \dots, B_t^d)$ be the standard d-dimensional Brownian motion equipped with its standard Brownian filtration $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $K_t = (K_t^1, \dots, K_t^d) \in \mathbb{R}^d$ be predictable with

$$\mathbb{E}\int_0^t |K_s|^2\, ds < \infty, \quad \text{for all} \quad t > 0.$$

For any two vectors $u, v \in \mathbb{R}^d$, we use $u \cdot v = \langle u, v\rangle$ to denote the standard Euclidean inner product on $\mathbb{R}^d$. Define the martingale given by the stochastic integral

$$M_t = M_0 + \int_0^t K_s \cdot dB_s, \text{ with quadrant variation } \langle M\rangle_t = \int_0^t |K_s|^2 ds. \tag{3.1}$$

We will often refer to martingales of this form as *Brownian martingales.* If N_t is another martingale given by a predictable process $H_t \in \mathbb{R}^d$, the covariation process of M and N is given by

$$\langle M, N\rangle_t = \int_0^t M_s \cdot K_s ds.$$

Definition 3.1. *Consider two martingales N_t and M_t as above. The martingale M_t is said to be subordinate to N_t if for all t, $\langle M\rangle_t \leq \langle N\rangle_t$. They are said to be orthogonal if $\langle M, N\rangle_t = 0$, for all t. Throughout the paper, $\|M\|_p = \sup_{t>0} \|M_t\|_p$.*

Theorem A ([5, 32])**.** *Suppose M and N are two martingales as above.*

(1) Suppose M is subordinate to N. Then

$$\|M\|_p \leq (p^* - 1)\|N\|_p, \quad 1 < p < \infty. \tag{3.2}$$

(2) Suppose M and N are orthogonal martingales and M is subordinate to N. Then

$$\|M\|_p \leq \cot\left(\frac{\pi}{2p^*}\right)\|N\|_p, \quad 1 < p < \infty. \tag{3.3}$$

Both inequalities are sharp.

Inequality (3.2) is a case of the celebrate inequalities of Burkholder [32] which have had multiple applications in different areas of analysis, probability and related fields. Burkholder inequality hold for Hilbert space-valued martingalesn with the same cosnatnt. The notion of subordinate orthogonal martingales was introduced in [5] where the inequality is proved. This too has Hilbert space-valued versions [24, Theorem A]. The main application in that paper is to prove the sharp inequality for Reisz transforms in (2.6). An advantage of the martingale approach is their applications to Riesz transforms on different geometric settings beyond $\mathbb{R}^d$, as the now large subsequent literature has shown. For parallel weak-type $(1, 1)$ inequalities, we refer the reader to [25] and references therein.

In addition to the literature already mention above, the interested reader is highly encouraged to consult [79] where many extensions and applications are presented as well as the survey article [77] where the Burkholder-Bellman function method, which is at the heart of the sharp martingale inequalities and their applications, is discussed.

In what follows, our goal is to present a martingale analogue of Cotlar's identity and to use it to derive (3.3) in the special case where $\langle M\rangle_t = \langle N\rangle_t$ and $p = 2^k$, $k \in \mathbb{N}$, paralleling Cotlar's original argument for the Hilbert transform.

With M and N as above, by Itô's formula

$$N_t M_t = \int_0^t M_s dN_s + \int_0^t N_s dM_s + \langle N, M\rangle_t \tag{3.4}$$
$$= \int_0^t M_s H_s \cdot dB_s + \int_0^t N_s K_s \cdot dB_s + \int_0^t H_s \cdot K_s ds.$$

Taking $M_t = N_t$ gives

$$M_t^2 = 2\int_0^t M_s dM_s + \langle M\rangle_t = 2\int_0^t M_s K_s \cdot dB_s + \int_0^t |K_s|^2 ds. \tag{3.5}$$

Let $A = (a_{ij})$ be a $d \times d$ matrix with real coefficients. Set $\|A\|^2 = \sup\{|Av|^2 : v \in \mathbb{R}^d, |v| = 1\}$. The martingale transform of M_t by A is defined as

$$A * M_t = \int_0^t AK_s \cdot dB_s. \tag{3.6}$$

Theorem 3.2. *Suppose A is a $d \times d$ matrix such that $Av \cdot v = 0$ for all $v \in \mathbb{R}^d$. Then*

$$|A * M_t|^2 = 2A * (M(A * M))_t + \int_0^t |AK_s|^2 ds - 2\int_0^t [M_s A^2 K_s] \cdot dB_s \tag{3.7}$$

Proof. Applying (3.4) to the product of M_t and $(A * M)_t$ gives

$$M_t A * M_t = \int_0^t A * M_s K_s \cdot dB_s + \int_0^t M_s AK_s \cdot dB_s + \int_0^t AK_s \cdot K_s ds$$
$$= \int_0^t \tilde{K} \cdot dB_s,$$

where $\tilde{K_s} = [A * M_s K_s + M_s AK_s]$. The assumption on A gives $AK_s \cdot K_s = 0$ and $\langle M, A * M\rangle_t, = 0$. Thus $M_t(A * M)_t$ is a martingale of the above form. Its martingale transform by A is

(3.8)
$$A * (M(A * M)_t = \int_0^t A\tilde{K_s} \cdot dB_s = \int_0^t [(A * M)_s AK_s + M_s A^2 K_s] \cdot dB_s.$$

Similarly applying (3.5) to $(A * M)_t$ gives

$$|A * M)_t|^2 = 2\int_0^t [A * M_s AK_s] \cdot dB_s + \int_0^t |AK_s|^2 ds. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$2A * (M(A * M)_t = [(A * M_t]^2 + 2\int_0^t [M_s A^2 K_s] \cdot dB_s - \int_0^t |AK_s|^2 ds.$$

Equivalently,

$$|A * M_t|^2 = 2A * M(A * M)_t - 2\int_0^t [M_s A^2 K_s] \cdot dB_s + \int_0^t |AK_s|^2 ds, \tag{3.10}$$

which is the claimed identity. □

Corollary 3.3. *Suppose A is as in the statement of the Theorem. Then,*

$$|A * M_t|^2 \le 2A * M(A * M)_t + \|A\|^2 |M_t|^2 - 2\int_0^t [M_s A^2 K_s + \|A\|^2 M_s K_s] \cdot dB_s.$$

Proof. Notice that by (3.10),

$$|(A * M)_t|^2 \le 2(A * (M(A * M))_t - 2\int_0^t [M_s A^2 K_s] \cdot dB_s + \|A\|^2 \int_0^t |K_s|^2 ds$$

Applying (3.5) with $Y_t = M_t$ gives

$$\int_0^t |K_s|^2 ds = M_t^2 - 2\int_0^t M_s K_s \cdot dB_s$$

and it follows that

$$\begin{aligned} |(A * M)_t|^2 &\le 2A * (M(A * M))_t - 2\int_0^t [M_s A^2 K_s] \cdot dB_s + \|A\|^2 [M_t^2 \\ &\quad - 2\int_0^t M_s K_s \cdot dB_s] \\ &= 2A * (M(A * M))_t + \|A\|^2 M_t^2 \\ &\quad - 2\int_0^t [M_s A^2 K_s + \|A\|^2 M_s K_s] \cdot dB_s, \end{aligned}$$

which gives competes the proof. ☐

4. Martingale transforms

4.1. Cotlar martingale transforms.

Definition 4.1. *Supposed $d = 2n$ is even and $A = (a_{ij})$ is a $d \times d$ real matrix. We will say A is a Cotlar matrix if it has the following properties: (1) $Av \cdot v = 0$ for all $v \in \mathbb{R}^d$ and (2) $A^2 = -I$, where I denotes the identity matrix. A martingale transform by a Cotlar matrix will be called a Cotlar martingale transform.*

Note that since $\det(A^2) = (\det(A))^2 = (-1)^d$, there are no such $d \times d$ matrices when d is odd.

The Cotlar martingale identity now follows from Theorem 3.2.

Corollary 4.2 (Cotlar's martingale identity)**.** *Suppose d is even and A is a Cotlar matrix. Let M_t be a martingale as in (3.1). Then*

$$|A * M_t|^2 = 2A * (M\,(A * M))_t + M_t^2. \tag{4.1}$$

Proof. Since A is a Cotlar matrix,

$$\int_0^t |AK_s|^2 ds - 2\int_0^t [M_s A^2 K_s] \cdot dB_s = \int_0^t |K_s|^2 ds + 2\int_0^t [M_s K_s] \cdot dB_s = M_t^2,$$

by (3.5). This completes the proof. ☐

Since $2A*(M\,(A*M))_t$ is a martingale, its expectation is 0. Thus $\mathbb{E}|A*M_t|^2 = \mathbb{E}|M_t|^2$. From this and (4.1) the exact same induction proof as in (2.3) gives

Theorem 4.3. *Suppose $d = 2n$ is even and A is a Cotlar matrix. Then,*

$$\|A * M_t\|_p \le \cot\left(\frac{\pi}{2p^*}\right)\|M_t\|_p, \quad p = 2^k \text{ or } p = \frac{2^k}{2^-1}, \quad k \in \mathbb{N}. \tag{4.2}$$

Remark 4.4. *For the applications to Riesz transforms on $\mathbb{R}^d$ we will need to work with $(d+1)\times(d+1)$ matrices. In order to make the connection to Cotlar matrices we need to restrict to odd dimensions so that $d+1$ is even.*

Here are some examples. When $d = 1$, the following Cotlar matrix

$$\mathbf{H}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.3}$$

gives the probabilisitic representation of the Hilbert transform on $\mathbb{R}$.

When $d = 3$ the following three 4×4 Cotlar matrices

(4.4)

$$\mathbf{H}_4^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \ \mathbf{H}_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ \mathbf{H}_4^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

give the probabilisitic representation of the three Riesz transforms: R_1, R_2, R_3 on $\mathbb{R}^3$, respectively. They also have the property that $\langle H_4^j v, H_4^k v\rangle = 0$, for all $j \neq k$ and all vectors $v \in \mathbb{R}^4$, which gives a complex structure on $\mathbb{C}^2$.

In general, suppose d is odd so that $d + 1 = 2n$. Then the following Cotlar matrix

$$\mathbf{H}_{2n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix} = \operatorname{diag}(\underbrace{\mathbf{H}_2, \mathbf{H}_2, \ldots, \mathbf{H}_2}_{n-\text{times}}), \tag{4.5}$$

corresponds to the first Riesz transform R_1 on $\mathbb{R}^d$. The other $(d-1)$ Riesz transforms arise from orthogonal rotations of this matrix.

4.2. **Conformal martingales.** Before we make the connection] to Riesz transforms on $\mathbb{R}^d$, we present two other (and more general) version of Theorem 4.3.

Definition 4.5 (Conformal martingale)**.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \ge 0\}, \mathbb{P})$ be a filtered probability space with the usual conditions; the filtration is right continuous and $\mathcal{F}_0$ contains all sets of probability 0.*

(i) A $\mathbb{C}$-valued continuous local martingale $Z = X + iY$ is called conformal *if*

$$\langle X\rangle_t = \langle Y\rangle_t \;\; \textit{and} \;\; \langle X, Y\rangle_t = 0, \;\; \textit{for all } t \geq 0.$$

(ii) For any $n \geq 1$, a $\mathbb{C}^n$-valued continuous local martingale $Z = (Z^1, \dots, Z^n)$ is called conformal if Z^j is conformal for every $j = 1, \dots, n$.

(iii) A $\mathbb{C}^n$-valued conformal local martingale is said to satisfy the orthogonality property if

$$\langle Z_t^k, Z_t^j\rangle = 0, \;\; \textit{for all } \; j, \; k. \tag{4.6}$$

Note that we always have (4.6) for $j = k$ for any $\mathbb{C}^n$-valued conformal martingale

Remark 4.6. *In many of the papers in the literature on sharp martingale inequalities and applications to singular integrals starting with [5], and [24], conformal martingales with the extra cross orthogonality condition (4.6) are simple called orthogonal martingales.*

Since all the martingales here are assumed to be L^p-bounded we will omit the local martingale terminology and simple use martingale.

Conformal martingales and their connections to analytic functions in the plane and $\mathbb{C}^n$, in general, have been extensively studied in the literature. For some of these literature and applications, see [47, 50, 72, 90].

Throughout the rest of this section we assume that all our martingales start at zero. Notice that if A is a Cotlar matrix the martingale $Z_t = M_t + iA * M_t$ on the Brownian filtration is a conformal martingale. The proof of (4.1) and that of Cotlar's original identity (2.1) using analytic functions immediately give a similar identity for $\mathbb{C}$-valued conformal martingales.

We start with a lemma connecting conformal martingales to analytic functions on $\mathbb{C}$. The lemma is a special case of Proposition 5.4 in [50, pg. 291] valid for holomorphic functions on $\mathbb{C}^n$. For our needs here it suffices to take of $f : \mathbb{C} \to \mathbb{C}$ given by $F(z) = z^2$. However, the proof of the case z^2 is really the same as the proof for the case z^k and for completeness we give the proof.

Lemma 4.7. *If $Z = X + iY$ is a $\mathbb{C}$-valued conformal martingale and $k \in \{1, 2, 3, \dots\}$. Then*

$$Z_t^{(k)} := (Z_t)^k$$

is again a continuous conformal martingale. Writing $Z_t^{(k)} = U_t + iV_t$, we have

$$\langle U\rangle_t = \langle V\rangle_t, \qquad \langle U, V\rangle_t = 0.$$

Moreover,

$$d\langle Z^{(k)}\rangle_t = k^2 |Z_t|^{2k-2} \, dA_t, \quad \textit{where} \quad A_t = \langle X\rangle_t = \langle Y\rangle_t.$$

Proof. Set $F(z) = z^k$. Since f is analytic we write $F(z) = f(x, y) = U(x, y) + iV(x, y)$, where U and V are conjugate harmonic functions satisfying the Cauchy-Riemann equations. It follows from Itô's formula that both $U_t = U(X_t, V_t)$ and

$V_t = V(X_t, Y_t)$ are martingales. Applying (3.4) and (3.5) with these two martingales gives:

$$d\langle U\rangle_t = |\nabla U|^2 dA_t, \quad d\langle V\rangle_t = |\nabla V|^2\, dA_t,$$
$$d\langle U, V\rangle_t = (U_x V_x + U_y V_y)\, dA_t.$$

By the Cauchy–Riemann equations,

$$|\nabla U|^2 = |\nabla V|^2$$
$$U_x V_x + U_y V_y = U_x(-U_y) + U_y(U_x) = 0$$

Thus $\langle U\rangle_t = \langle V\rangle_t$ and $\langle U, V\rangle_t = 0$. Recalling that $|\nabla U(z)|^2 = |F'(z)|^2$ completes the proof of the Lemma. □

Definition 4.8. *For any* $t > 0$ *and* $1 < p < \infty$*, define*

$$\beta_p := \sup\left\{\frac{\|Y_t\|_p}{\|X_t\|_p}\right\}, \tag{4.7}$$

where the sup *is taken over all pairs* (X, Y) *of martingales such that* $Z = X + iY$ *is a conformal martingale.*

Since $\langle X\rangle = \langle Y\rangle$ for all conformal pairs, $\beta_2 = 1$. The fact that β_p is finite follows from any one of the multiple versions of the the classical Burkholder-Gundy inequalities.

Our interest here is in showing that

$$\beta_p \le \cot\left(\frac{\pi}{2p}\right), \quad p = 2^k,\ k \in \mathbb{N}.$$

By Lemma 4.7,

$$Z^2 = X^2 - Y^2 + 2iXY = U + iV \tag{4.8}$$

is a $\mathbb{C}$-valued conformal martingale with real and imaginary parts given by

$$U_t = X_t^2 - Y_t^2, \quad V_t = 2X_tY_t,$$

$$X_t^2 - Y_t^2 = 2\int_0^t \big(X_s\, dX_s - Y_s\, dY_s\big), \quad 2X_tY_t = 2\int_0^t \big(X_s\, dY_s + Y_s\, dX_s\big).$$

In this setting,

$$Y_t^2 = X_t^2 - \Re Z_t^2 \tag{4.9}$$

is the needed Cotlar identity.

Note that without conformality we would have

$$Z_t^2 = 2\int_0^t Z_s dZ_s + \langle Z\rangle_t,$$

with $\langle Z_t\rangle \neq 0$ and Z^2 would not be a martingale.

Theorem 4.9. *Let $Z = X + iY$ be a conformal martingale. Then*

$$\|Y\|_p \leq \cot\left(\frac{\pi}{2p}\right)\|X\|_p, \quad p = 2^k, \quad n \in \mathbb{N}. \tag{4.10}$$

Proof. Suppose we can show that for every $p > 1$,

$$\beta_{2p} \leq \beta_p + \sqrt{1 + \beta_p^2}, \tag{4.11}$$

where β_p is the constant in (4.7). Since $\mathbb{E}(Y_t^2) = \mathbb{E}\langle Y\rangle_t = \mathbb{E}\langle X\rangle_t) = \mathbb{E}(X_t^2)$, it follows that $\beta_2 = 1$. The induction argument as before gives that for $p = 2^k$, $\beta_p \leq \cot\left(\frac{\pi}{2p}\right)$, and (4.10) follows.

To show (4.11), applying the definition of β_p to the pair (U, V) in (4.8) we have

$$\|X_t^2 - Y_t^2\|_p \leq \beta_p \, \|2X_tY_t\|_p.$$

Writing

$$Y_t^2 = X_t^2 - (X_t^2 - Y_t^2)$$

and applying the Minkowski and Cauchy–Schwarz inequalities, exactly as in Cotlar's proof, we obtain

$$\begin{aligned}
\|Y_t\|_{2p}^2 = \|Y_t^2\|_p &\leq \|X_t^2\|_p + \beta_p\|2X_tY_t\|_p \\
&\leq \|X_t^2\|_p + 2\beta_p\|Y_t\|_{2p}\|X_t\|_{2p} \\
&\leq \|X_t^2\|_p + 2\beta_p\beta_{2p}\|X_t\|_{2p}^2 \\
&= \|X_t\|_{2p}^2 + 2\beta_p\beta_{2p}\|X_t\|_{2p}^2.
\end{aligned}$$

Hence,

$$\frac{\|Y_t\|_{2p}^2}{\|X_t\|_{2p}^2} \leq 1 + 2\beta_p\beta_{2p},$$

which again together with the definition of β_p gives

$$\beta_{2p}^2 \leq 1 + 2\beta_p\beta_{2p}.$$

This competes the proof of (4.11) and (4.10) follows.

□

The inequality (4.10) is a especial ase of the inequality in [5] valid for $1 < p < \infty$. We end this section by pointing out the version of Theorem 4.9 for $\mathbb{C}^n$-valued conformal martingales for any $n \geq 1$. This directly follows from the next lemma.

Lemma 4.10. *Suppose $Z = (Z^1, \ldots, Z^n) \in \mathbb{C}^n$ is a conformal martingale with the orthogonality property (4.6). Set $Z^k = X^k + iY^k$. Then*

$$\Gamma_t := Z_t \cdot Z_t = \sum_{k=1}^{n}(Z_t^k)^2 = \sum_{k=1}^{n}[(X^k)^2 - (Y^k)^2 + 2iX^kY^k] = U_t + iV_t$$

is a complex-valued conformal martingale with

$$U_t = \|X_t\|_{\ell^2}^2 - \|Y_t\|_{\ell^2}^2 \tag{4.12}$$

and

$$V_t = 2X_t \cdot Y_t = 2\sum_{k=1}^{n}\int_0^t \big(X_s^k\, dY_s^k + Y_s^k\, dX_s^k\big) \tag{4.13}$$
$$= 2\int_0^t (X_s \cdot dY_s + Y_s \cdot dX_s).$$

Proof. Since the function $F(z) = z_1^2 + \cdots + z_k^2$ is holomorphic, it follows from Itô's formula that Γ is a complex-valued conformal martingale. This is again a special case of Proposition 5.4 in [50, pg. 291] which states that if $F : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and $Z = (Z^1, \ldots, Z^n)$ is a $\mathbb{C}^n$-valued conformal martingale satisfying (4.6), then $F(Z)$ is a $\mathbb{C}$-valued conformal martingale.

However, verifying $F(Z)$ is a conformal martingale for our function F is very simple without appealing to the general result. Indeed, by Lemma 4.7, $(Z^k)^2$ is a martingale and hence so is their sum Γ. It remains to show that Γ is conformal. Computing the covariance and using (4.6),

$$\langle\Gamma\rangle_t = \langle\sum_{j=1}^{n}(Z^j)^2,\, \sum_{k=1}^{n}(Z^k)^2\rangle_t = \sum_{j,k=1}^{n}\langle(Z^j)^2,\, (Z^k)^2\rangle_t$$
$$= 4\sum_{j,k=1}^{n} Z_t^j\, Z_t^k\, \langle Z^j\, Z^k\rangle_t.$$
$$= 0.$$

□

Remark 4.11. *For a concrete example of a $\mathbb{C}^n$, $n = 2d$, valued conformal martingales with the orthogonal property that arises from a Beurling-Ahlfors operators on $\mathbb{R}^d$, see (6.30) and the discussions that follows.*

From Lemma 4.10 and the same proof as that of Theorem 4.9 we have the vector-valued version.

Theorem 4.12. *Let $Z = X + iY$ conformal $\mathbb{C}^n$-valued martingale with the orthogonality property. Then for $p = 2^k, k \in \mathbb{N}$,*

$$\|\, \|Y\|_{\ell^2} \|_p \le \cot\left(\frac{\pi}{2p}\right) \|\, \|X\|_{\ell^2}\, \|_p.$$

4.3. **Riesz transforms as projections of Cotlar martingale transforms.** Before we return to the Cotlar-Hilbert martingale transforms, we explain how the martingale transform on $\mathbb{R}^d$ arise as projections operators, equivalently conditional expectations, of the martingale transforms. This is the celebrated work of Gundy-Varopoulos [56] that has been so extensively used in the literature. Here, we use the notation in [6, 8]. Let $B_t = (X_t, Y_t) = (X_t^1, \ldots, X_t^d, Y_t)$ be $(d+1)$ dimensional Brownian motion in the upper half-space of $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0, \infty)$ starting with the Lebesgue measure on the hyperplane $\{(x, y) : x \in \mathbb{R}^d, y = a\}$, $a > 0$.

Let $\tau = \inf\{t > 0 : Y_t = 0\}$ be its exit time. We identify $B_\tau = (X_\tau, 0)$ with X_τ. If $\mathbb{P}_{(x,a)}$ is the probability measure associated with $B_t = (X_t, Y_t)$ starting at the point (x, a) with $x \in \mathbb{R}^d$ and $a > 0$, define the measure $\mathbb{P}^a$ by

$$\mathbb{P}^a(B_{t\wedge\tau}) \in \Theta) = \int_{\mathbb{R}^d} \mathbb{P}_{(x,a)}(B_{t\wedge\tau} \in \Theta) d\mu(x),$$

for any Borel set $\Theta \in \mathbb{R}^d \times \mathbb{R}^+$. In particular, for any Borel set $\Theta \subset \mathbb{R}^d$, $\mathbb{P}_y(X_\tau \in \Theta) = m(\Theta)$, where m is the Lebesgue measure on $\mathbb{R}^d$. Hence,

$$\mathbb{E}^a(f(X_\tau)) = \int_{\mathbb{R}^d} f(x) d\mu(x). \tag{4.14}$$

In the same way, by independence, the transition probability of $\{B_t; \tau > t\}$ is the product of the heat kernel in $\mathbb{R}^d$ and the heat kernel for the half-line $(0, \infty)$. Integrating away the heat kernel in the x-variable (since the Brownian motion has the Lebesgue measure as its initial distribution) and computing the Green's function for the half-line gives

$$\mathbb{E}^a \int_0^\tau F(B_s)\, ds = \int_0^\infty \mathbb{E}^a[F(B_s); \tau > t] = 2\int_0^\infty \int_{\mathbb{R}^d} (y \wedge a)\, F(x, y)\, dx\, dy, \tag{4.15}$$

for all nonnegative functions F on $\mathbb{R}^{d+1}_+$. Both (4.14) and (4.15) continue to hold for those f and F for which the integrals are finite.

At this point we could apply the martingale inequalities with respect to the probability measure of Brownian motion starting at the (x, a) and then let $a \to \infty$ to get our results for their projections. This is done in [6]. Here we will proceed as in the original paper of Gundy and Varopoulos [56] where they used a time-reversal argument to let $a \to \infty$ and construct a filtered probability space and a process $\{B_t = (X_t, Y_t)\}$ indexed by $t \in (-\infty, 0]$, which they called the *background radiation process.* Heuristically speaking, the paths of B_t are Brownian paths which originate from "$\{a = \infty\}$" at time $t = -\infty$ and exit $\mathbb{R}^{d+1}_+$ at time $t = 0$ with Lebesgue measure as their distribution. Letting $\mathbb{E}$ be the expectation with respect to the measure associated with the background radiation process, the identities (4.14) and (4.15) become, respectively,

$$\mathbb{E}f(B_0) = \int_{\mathbb{R}^d} f(x)\, dx \tag{4.16}$$

and

$$\mathbb{E} \int_{-\infty}^0 F(B_s)\, ds = 2\int_0^\infty \int_{\mathbb{R}^d} yF(x, y)\, dx\, dy. \tag{4.17}$$

For $f \in C_0^\infty(\mathbb{R}^d)$, let $U_f(x, y) = P_y f(x)$ be the convolution of f with the Poisson kernel

$$P_y(x) = \frac{c_d\, y}{(|x|^2 + |y|^2)^{\frac{d+1}{2}}},$$

where c_d is the constant as in the definition of the Riesz transforms (2.4). We set

$$\begin{aligned}\nabla U_f(x,y) &= (\partial_y U_f(x,y), \partial_{x_1} U_f(x,y), \dots, \partial_{x_d} U_f(x,y)) \\ &= (\partial_y P_y f(x), \partial_{x_1} P_y f(x), \dots, \partial_{x_d} P_y f(x),)\end{aligned}$$

and

$$dB_s = \left(dY_s, dX_s^1, \dots, dX_s^d\right).$$

Apply Itô's formula to get the martingale

$$M_t^f = U_f(X_t, Y_t) = \int_{-\infty}^t \nabla U_f(X_s, Y_s) \cdot dB_s, \quad t \in (-\infty, 0).$$

Note that

$$M_0^f = f(B_0) = \int_{-\infty}^0 \nabla U_f(X_s, Y_s) \cdot dB_s. \tag{4.18}$$

For any $(d+1) \times (d+1)$ matrix A define the martingale transform

$$A * M_t^f = \int_{-\infty}^t A \nabla U_f(X_s, Y_s) \cdot dB_s, \quad t \in (-\infty, 0]$$

and its projection operator (conditional expectation) on $\mathbb{R}^d$ by

(4.19)

$$T_A f(x) = \mathbb{E}[A * M_0^f \,\big|B_0 = (x,0)] = \mathbb{E}\left(\int_{-\infty}^0 A \nabla U_f(X_s, Y_s) \cdot dB_s \,\big|X_0 = x\right).$$

Since the conditional expectation is a contraction on L^p for any $1 < p < \infty$ and the distribution of B_0 is the Lebesgue measure, we immediately have

$$\|T_A f\|_p \leq \left(\mathbb{E}|A * M_0^f|^p\right)^{1/p} = \|A * M_0^f\|_p \;\; 1 < p < \infty. \tag{4.20}$$

From this and Theorem 4.3 we have

Corollary 4.13. *Suppose d is odd and A is a $(d+1) \times (d+1)$ Cotlar matrix. Then*

$$\|T_A f\|_p \leq \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p, \;\; p = 2^k \text{ or } p = \frac{2^k}{2^n - 1}, n \in \mathbb{N}. \tag{4.21}$$

Remark 4.14. *An important distinction for the case $d = 1$ comes from the fact that the Cauchy-Riemann equations give*

$$\mathbf{H}_2 \nabla U_f = \nabla U_{Hf}.$$

Consequently,

$$\mathbf{H}_2 * M_0^f = M_0^{Hf} = Hf(B_0). \tag{4.22}$$

Thus the conditional expectation plays no role in this case and equality holds in (4.20).

For any $d > 1$, consider the $(d+1) \times (d+1)$ matrices $\mathbf{R}_j$, $j = 1, 2, \ldots, d$, defined by

$$\mathbf{R}_j = [a^j_{lm}] = \begin{cases} 1, & l = 1, m = j+1 \\ -1, & l = j+1, m = 1 \\ 0, & \text{otherwise}, \end{cases} \tag{4.23}$$

and for $j, k = 1, 2, \ldots, d,\ j \neq k$, the matrices $\mathbf{R}_{(j,k)}$ defined by

$$\mathbf{R}_{(j,k)} = \left[a^{(j,k)}_{\ell m}\right] = \begin{cases} -1, & \ell = k+1,\ m = j+1, \\ -1, & \ell = j+1,\ m = k+1, \\ 0, & \text{otherwise}. \end{cases}$$

Example 4.15. *For example, with $d = 3$, we have the following:*

(4.24)

$$\mathbf{R}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0. \end{bmatrix}$$

and

$$\mathbf{R}_{(1,2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R}_{(1,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \mathbf{R}_{(2,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

In Addison, $\mathbf{R}_{(1,2)} = \mathbf{R}_{(2,1)}$, $\mathbf{R}_{(1,3)} = \mathbf{R}_{(3,1)}$ and $\mathbf{R}_{(2,3)} = \mathbf{R}_{(3,2)}$.

The above matrices can be decompose as the sum of two matrices in the form of

$$\mathbf{R}_j = \mathbf{R}^1_j + \mathbf{R}^2_j, \text{ and } \mathbf{R}_{(j,k)} = \mathbf{R}^1_{(j,k)} + \mathbf{R}^2_{(j,k)}$$

where,

$$\mathbf{R}^1_j = \begin{cases} 1, & l = 1, m = j+1 \\ 0, & \text{otherwise}, \end{cases}, \quad \mathbf{R}^2_j = \begin{cases} -1, & l = j+1, m = 1 \\ 0, & \text{otherwise}, \end{cases} \tag{4.25}$$

and

$$\mathbf{R}^1_{(j,k)} = \begin{cases} -1, & \ell = k+1,\ m = j+1, \\ 0, & \text{otherwise}, \end{cases}, \tag{4.26}$$

$$\mathbf{R}^2_{(j,k)} = \begin{cases} -1, & \ell = j+1,\ m = k+1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.27}$$

For $\{j, k = 1, \ldots, d\}$, the second order Riesz transforms are the Fourier multiplier operators with

$$\widehat{R_{(j,k)}f}(\xi) = \frac{-\xi_1\xi_2}{|\xi|^2}\widehat{f}(\xi).$$

Lemma 4.16. *For all $f \in C_0^\infty(\mathbb{R}^d)$ the following hold.*

(i) For all $j \in \{1, \dots, d\}$,

$$T_{\mathbf{R}_j^1} f = T_{\mathbf{R}_j^2} f = \frac{1}{2} R_j f. \quad \textit{Hence} \quad T_{\mathbf{R}_j} f = R_j f.$$

(ii) For all $j \neq k \in \{1, \dots, d\}$,

$$T_{\mathbf{R}^1_{(j,k)}} f = T_{\mathbf{R}^2_{(j,k)}} f = \frac{1}{2} R_{(j,k)} f. \quad \textit{Hence} \quad T_{\mathbf{R}_{(j,k)}} f = R_{(j,k)} f.$$

Proof. This follows from Gundy-Varopoulos [56]. Here we give the proof of (i) following the computation in [8, pg. 817]. The proof of (ii) is exactly the same. Recall that

$$\widehat{P_y f}(\xi) = e^{-2\pi y|\xi|} \widehat{f}(\xi) \text{ and } \widehat{\partial_{x_j} f}(\xi) = -2i\pi \xi_j \widehat{f}(\xi)$$

Let g be another smooth function of compact support. Set

$$N = g(B_0) = \int_{-\infty}^0 \nabla U_g(X_s, Y_s) \cdot dB_s$$

and

$$M = \mathbf{R}_j^1 * M_0^f = \int_{-\infty}^0 \mathbf{R}_j^1 \nabla U_f(X_s, Y_s) \cdot dB_s = \int_{-\infty}^0 \partial_{x_j} U_f(X_s, Y_s) dY_s.$$

By (3.4) and (4.17) we have

$$\begin{aligned}
\mathbb{E}[NM] = \mathbb{E}\langle N, M\rangle &= \mathbb{E} \int_{-\infty}^0 \partial_{x_j} U_f(X_s, Y_s) \partial_y U_g(x, y) ds \\
&= \int_0^\infty \int_{\mathbb{R}^d} 2y\, \partial_{x_1} U_f(x, y) \partial_y U_g(x, y) dx\, dy.
\end{aligned}$$

This gives,

$$\begin{aligned}
\int_{\mathbb{R}^d} g(x) T_{\mathbf{R}_j^1} f(x) dx &= \mathbb{E}[g(B_0) T_{\mathbf{R}_j^1} f(B_0)] \\
&= \mathbb{E}\left(\mathbb{E}\left(g(B_0) \int_{-\infty}^0 \partial_{x_j} U_f(X_s, Y_s) dY_s\right) \Big| B_0\right) \\
&= \mathbb{E}[NM] = \mathbb{E} \int_{-\infty}^0 \partial_{x_j} U_f(X_s, Y_s) \partial_y U_g(x, y) ds \\
&= \int_0^\infty \int_{\mathbb{R}^d} 2y\, \partial_{x_1} U_f(x, y) \partial_y U_g(x, y) dx\, dy \\
&= 2 \int_0^\infty y \int_{\mathbb{R}^d} \partial_{x_j} \widehat{U_f(\xi, y)} \overline{\partial_y \widehat{U_f(\xi, y)}}, d\xi, \\
&= 8\pi^2 \int_{\mathbb{R}^d} i\xi_j |\xi|\, \widehat{f}(\xi)\, \overline{\widehat{g}}(\xi) \left(\int_0^\infty y e^{-4\pi y|\xi|}\, dy\right) d\xi \widehat{R_j f}(\xi)\, \overline{\widehat{g}}(\xi)\, d\xi \\
&= \frac{1}{2} \int_{\mathbb{R}^d} R_j f(x)\, g(x)\, dx.
\end{aligned}$$

This shows that $T_{\mathbf{R}_j^1} f = \frac{1}{2} R_j f$ and competes the proof of (i). The proof of (ii) is exactly the same. $\square$

Remark 4.17. *The use of the background radiation process can be entirely avoided by defining the operators*

$$\mathcal{T}_A^a f(x) = \mathbb{E}^a \left(\int_0^\tau A \nabla U_f(X_s, Y_s) \cdot dB_s \big| X_\tau = x \right).$$

Taking $A = \mathbf{R}_j$ and integrating against another smooth function as above and letting $a \to \infty$ shows that $\mathcal{T}_{\mathbf{R}_j}^a f \to R_j f$ in L^2. Since all the L^p-bounds of the operators $\mathcal{T}_{\mathbf{R}_j}^a$ are independent of a, they remain valid for R_j. For more details on this construction, see [6].

The lemma immediately gives the following

Corollary 4.18. *Consider the $(d+1) \times (d+1)$ matrix*

$$\tilde{\mathbf{R}}_{(j,k)} = \left[\tilde{a}_{\ell m}^{(j,k)}\right] = \begin{cases} 1, & \ell = j+1,\ m = k+1, \\ -1, & \ell = k+1,\ m = j+1 \\ 0, & \textit{otherwise}, \end{cases}$$

then

$$T_{\tilde{\mathbf{R}}_{(j,k)}} f = \frac{1}{2} \left(R_{(j,k)} f - R_{(k,j)} f \right) = 0. \tag{4.28}$$

Theorem 4.19. *Suppose $d > 2$ is odd so that $d + 1 = 2n$ for and integer $n > 1$. Let R_1 be the first Riesz transform on $\mathbb{R}^d$. Consider the Cotlar matrix $\mathbf{H}_{2n}$ given in (4.5). Then the projection of the Cotlar martingale transform $\mathbf{H}_{2n} * M_t^f$ is $R_1 f$. That is,*

$$T_{\mathbf{H}_{2n}} f = R_1 f \tag{4.29}$$

and

$$\|R_1 f\|_p \le \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p,\ p = 2^k \textit{ or } p = \frac{2^k}{2^k - 1}, k \in \mathbb{N}. \tag{4.30}$$

Proof. Observe that the matrix H_{2n} is the sum of the elementary skew symmetric matrices generating the 2×2 blocks on the index pairs $(1,2)$, $(3,4)$, $\dots$, $(2n-1, 2n)$. For $k = 1, \dots, n-1$, define as above the $(d+1) \times (d+1)$ matrix $\widetilde{\mathbf{R}}_{(2k,2k+1)}$ by

$$\widetilde{\mathbf{R}}_{(2k,2k+1)} = \left[\tilde{a}_{\ell m}^{(2k,2k+1)}\right] = \begin{cases} 1, & \ell = 2k+1, m = 2k+2), \\ -1, & \ell = (2k+2, m = 2k+1), \\ 0, & \text{otherwise}. \end{cases}$$

That is, $\widetilde{\mathbf{R}}_{(2k,2k+1)}$ is zero except for a single 2×2 skew-symmetric block acting on rows and columns $2k+1$ and $2k+2$. Then $\mathbf{H}_{2n} = \mathbf{R}_1 + \sum_{k=1}^{n-1} \widetilde{\mathbf{R}}_{(2k,2k+1)}$

and

$$\mathbf{H}_{2n} * M_t^f = \mathbf{R}_1 * M_t^f + \sum_{k=1}^{n-1} \widetilde{\mathbf{R}}_{(2k,2k+1)} * M_t^f. \tag{4.31}$$

Taking conditional expectation and employing (4.28) we obtain

$$T_{\mathbf{H}_{2n}} f(x) = R_1 f(x) + \sum_{k=1}^{n-1} T_{\widetilde{\mathbf{R}}_{(2k,2k+1)}} f(x) = R_1 f(x). \tag{4.32}$$

The bound in (4.30) follows from Corollary 4.13, completing the proof of the theorem. □

Example 4.20. *Suppose $d = 3$ and consider the Cotlar matrices $\mathbf{H}_4^1, \mathbf{H}_4^2, \mathbf{H}_4^3$ given in (4.4). The above construction gives:*

(1) $T_{\mathbf{H}_4^1} = R_1 + \frac{1}{2}(R_{(2,3)} - R_{(3,2)}) = R_1$
(2) $T_{\mathbf{H}_4^2} = R_2 + \frac{1}{2}(R_{(1,3)} - R_{(3,1)}) = R_2$
(3) $T_{\mathbf{H}_4^3} = R_3 + \frac{1}{2}(R_{(1,2)} - R_{(2,1)}) = R_3$

Remark 4.21. *Extending the L^p bound from odd to even dimensions presents no problem. Recall that by de Leeuw's extension theorem (*[39], [54, Theorem 2.5.15]*) the Riesz transforms on $\mathbb{R}^{d_2}$ are Fourier extensions of those on $\mathbb{R}^{d_1}$, $d_2 > d_1$. Hence the upper bound for $\|R_1\|_{p\to p}$ in the Corollary holds for all $d > 1$. Of course, we already knew that but here we have shown it in the case of powers of 2 directly from the Cotlar identity.*

Remark 4.22. *By a rotation of the matrix $\mathbf{H}_{2n}$, as in the example for $d = 3$, we can verify that R_j, $j = 2, \dots, d$, are projection of a Cotlar martingale transforms and hence we have the same bound for R_j as for R_1. However, if the only thing we want is to get the same L^p norm for R_j as that of R_1, this is immediate. Indeed, for $j = 2, \dots, d$, let ρ be orthogonal matrix $\rho \in O(d)$ (the orthogonal group in $\mathbb{R}^d$) with $\rho e_1 = e_j$ and define $(U_\rho f)(x) = f(\rho^{-1}x)$. Then $R_j = U_\rho^{-1} R_1 U_\rho$. Consequently, $\|R_j f\|_p = \|R_1(f \circ \rho^{-1})\|_p$ and hence $\|R_j\|_{p\to p} = \|R_1\|_{p\to p}$.*

4.4. **On a theorem and a problem of R. Durrett.** As we showed above, for any $d > 1$ the $(d+1) \times (d+1)$ matrices $\mathbf{R}_j$, $j = 1, \dots, d$ in (4.23) have the property that $T_{\mathbf{R}_j} = R_j$. That is, the Riesz transforms R_j are projections (conditional expectations) of the martingale transforms by the matrices $\mathbf{R}_j$ applied to the Brownian martingales obtained by composting the harmonic extensions of functions with Brownian motion in the upper half-space of $\mathbb{R}^{d+1}$. Motivated by Davis [38] probabilistic proof of Stein and Wiess [88] theorem that the distribution of the the conjugate function on the circle, equivalently of Hilbert transform, of an indicator function of a Lebesgue measurable set depends only on measure of the set, Durrett proved the following

Theorem A (R. Durret [47]). *Let $B \in \mathcal{F}_\infty$, $0 < \mathbb{P}(B) < 1$. There are constants C and γ which only depend on $\mathbb{P}(B)$ and the matrices so that*

$$\mathbb{P}\left(\sup_j \sup_t |\mathbf{R}_j * 1_B| > y\right) \geq Ce^{-\gamma y}. \tag{4.33}$$

Problem 4.23. *The validity of such an inequality for the Riesz transforms is raised in* [47, p. 34]. *To the best of our knowledge this problem remains open.*

As we have already seen, when d is odd, equivalent $d+1 = 2n$, the Riesz transforms on $\mathbb{R}^d$ are given by the projection operators $T_{\mathbf{H}_1}, \ldots, T_{\mathbf{H}_d}$ where the matrices $\mathbf{H}_1, \ldots, \mathbf{H}_d$ are $2n \times 2n$, $d+1 = 2n$, and satisfy

(1) **Cotlar property**

$$x \cdot \mathbf{H}_j x = 0, \quad |\mathbf{H}_j x| = |x|, \quad \text{for all } x \in \mathbb{R}^{2n}.$$

(2) **Mutual orthogonality**

$$(\mathbf{H}_i x) \cdot (\mathbf{H}_j x) = 0 \quad i \neq j, \quad \text{for all } x \in \mathbb{R}^{2n}.$$

Using these matrices a sharp version of (4.33) with an explicit constants is possible. This fact was already indicated in Durrett's paper following Davis's argument for the Stein-Weiss result. Here we also the compute the exact distribution of the martingale transform.

Towards that end, let $B_t = (B_t^1, \ldots, B_t^{2n})$ be the standard $2n$-dimensional Brownian motion equipped with the standard Brownian filtration $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{E} \in \mathcal{F}_\infty$ with $0 < \mathbb{P}(\mathcal{E}) < 1$. Consider the martingale

$$M_t = \mathbb{E}(1_{\mathcal{E}} \mid \mathcal{F}_t)$$

which can be written as

$$M_t := \mathbb{P}(\mathcal{E}) + \int_0^t K_s \cdot dB_s, \tag{4.34}$$

where $K_s \in \mathbb{R}^{2n}$ predicable. To simplify notation set $Y_t^j = (\mathbf{H}_j * 1_{\mathcal{E}})_t$, the martingale transforms by the matrices $\mathbf{H}_j$, $j = 1, \ldots, 2n$, and $Y_t^0 = M_t - \mathbb{P}(\mathcal{E})$. By properties (1) and (2) the martingale

$$Z_t := (Y^0, Y_t^1, \ldots, Y_t^{2n}) = (Y_t^0, Y_t^1, \ldots, Y_t^{2n}) \in \mathbb{R}^{2n+1}$$

is an orthogonal martingale. That is, $\langle Y^j, Y^k \rangle_t = 0$, for all $j \neq k$ and $d\langle Y^j, Y^j \rangle_t = |K_t|^2 dt$. Hence

$$d\langle Z \rangle_t = |K_t|^2 I_{2n+1}\, dt.$$

By Dambis-Dubins-Schwarz theorem (see [47, (1), p. 78]) there exists an $(2n+1)$-dimensional Brownian motion $\widetilde{B}_t$ such that $Z_t = \widetilde{B}_{\langle X \rangle_t}$. Moreover, if we set $\tau := \langle X \rangle_\infty$, then

$$(Y_\infty^1, \ldots, Y_\infty^{2n}) \stackrel{d}{=} (\widetilde{B}_\tau^{(1)}, \ldots, \widetilde{B}_\tau^{(2n)}),$$

and since $M_\infty = 1_{\mathcal{E}} \in \{0, 1\}$, τ is the first exit time of one dimensional Brownian motion from the interval $(-\mathbb{P}(\mathcal{E}), 1 - \mathbb{P}(\mathcal{E}))$.

Lemma 4.24. *Set $\beta = \mathbb{P}(\mathcal{E})$. Let τ be the first exit time of planar Brownian motion (B_t^1, B_t^2) started at $(0,0)$ from the strip*

$$S = (-\beta,\, 1-\beta) \times \mathbb{R}.$$

Then the distribution of the vertical coordinate B_τ^2 has density

$$f_\beta(\lambda) = \frac{\sin(\pi\beta)\cosh(\pi\lambda)}{\sinh^2(\pi\lambda) + \sin^2(\pi\beta)}, \qquad y \in \mathbb{R}. \tag{4.35}$$

which in particular depends only on the probability of the event $\mathcal{E}$.

These type of lemmas are usually proved by conformally mapping the upper half-space, whose Poisson kernel is

$$\frac{1}{\pi}\frac{y}{(|x|^2 + y^2)},$$

onto the strip by the exponential map $F(z) = e^{i\pi z}$. We leave this to the reader as an easy exercise in basic complex analysis. For examples of these type of applications, see [38, Theorem 3] and [64, Lemma 2,1].

Applying the Lemma with the pair (Y_t^0, Y_t^j), $j \geq 1$, it follows that

(4.36)

$$P(|Y_\infty^{(j)}| > \lambda) = \mathbb{P}(|\widetilde{B}_\tau^{(j)}| > \lambda) = 2\int_\lambda^\infty f_\beta(x)\, dx = \frac{2}{\pi}\arctan\left(\frac{\sin(\pi\beta)}{\sinh(\pi\lambda)}\right).$$

For the last equality, we make the substitution $t = \sinh(\pi x)$, $dt = \pi\cosh(\pi x)\, dx$ and get

$$2\int_\lambda^\infty f_\beta(x)\, dx = \frac{2\sin(\pi\beta)}{\pi}\int_{\sinh(\pi y)}^\infty \frac{dt}{t^2 + (\sin(\pi\beta))^2} = \frac{2}{\pi}\arctan\left(\frac{\sin(\pi\beta)}{\sinh(\pi\lambda)}\right).$$

Proposition 4.25. *For every $0 < \beta < 1$ and every $\lambda > 0$,*

$$\begin{aligned} \frac{2\arctan 2}{\pi}\sin(\pi\beta)e^{-\pi\lambda} &\leq \frac{2}{\pi}\arctan\left(\frac{\sin(\pi\beta)}{\sinh(\pi\lambda)}\right) \\ &\leq \min\left\{1, \frac{8}{\pi}\sin(\pi\beta)e^{-\pi\lambda}\right\}. \end{aligned} \tag{4.37}$$

Proof. To simplify notation, set $a = \sin(\pi\beta) \in (0,1)$ and $t = \pi\lambda > 0$.

Upper bound: Since $\arctan(x) \leq \pi/2$ for $x \geq 0$,

$$\frac{2}{\pi}\arctan\left(\frac{a}{\sinh t}\right) \leq 1. \tag{4.38}$$

Moreover, for $t > 0$, $\sinh t > 0$, so $\frac{a}{\sinh t} > 0$ and, using $\arctan(x) \leq x$, for $x \geq 0$, we have

$$\frac{2}{\pi}\arctan\left(\frac{a}{\sinh t}\right) \leq \frac{2}{\pi}\frac{a}{\sinh t}.$$

If $t \geq \frac{1}{2}\log 2$, then

$$\sinh t = \frac{e^t - e^{-t}}{2} = \frac{e^t}{2}(1 - e^{-2t}) \geq \frac{e^t}{4},$$

and hence

$$\frac{a}{\sinh t} \le 4ae^{-t}.$$

Therefore,

$$\frac{2}{\pi} \arctan\Big(\frac{a}{\sinh t}\Big) \le \frac{8}{\pi} ae^{-t}.$$

Combining this estimate with (4.38) gives

$$\frac{2}{\pi} \arctan\Big(\frac{a}{\sinh t}\Big) \le \min\Big\{1, \frac{8}{\pi} ae^{-t}\Big\}.$$

which is the righthand side of (4.37)

Lower bound. For all $t \ge 0$,

$$\sinh t = \frac{e^t - e^{-t}}{2} \le \frac{e^t}{2},$$

so

$$\frac{a}{\sinh t} \ge 2ae^{-t}.$$

Since $\arctan$ is increasing,

$$\arctan\Big(\frac{a}{\sinh t}\Big) \ge \arctan\big(2ae^{-t}\big).$$

For $0 \le x \le 2$, the function $u \mapsto \arctan x/x$ is decreasing, hence

$$\arctan x \ge \frac{\arctan 2}{2} x.$$

Since $0 \le 2ae^{-t} \le 2$, we obtain

$$\arctan\big(2ae^{-t}\big) \ge \arctan 2\, ae^{-t}.$$

Therefore,

$$\frac{2}{\pi} \arctan\Big(\frac{a}{\sinh t}\Big) \ge \frac{2\arctan 2}{\pi} ae^{-t},$$

proving the lefthand side of (4.37) and the claim. □

From (4.36) and (4.37), it follows that for all $\lambda > 0$,

$$\frac{2\arctan 2}{\pi} \sin(\pi\beta)e^{-\pi\lambda} \le \mathbb{P}(|\mathbf{H}_j * 1_{\mathcal{E}}| > \lambda) \le \min\Big\{1, \frac{8}{\pi} \sin(\pi\beta)e^{-\pi\lambda}\Big\}.$$

By the reflection principle for Brownian motion, we have

$$\mathbb{P}(|\widetilde{B}_\tau| > \lambda) \le \mathbb{P}\Big(\sup_{0\le s\le\tau} |\widetilde{B}_\tau| > \lambda\Big) \le 2\,\mathbb{P}(|\widetilde{B}_\tau| > \lambda).$$

$$\frac{2\arctan 2}{\pi} \sin(\pi\beta)e^{-\pi\lambda} \le \mathbb{P}\left(\sup_{t>0} |(\mathbf{H}_j * 1_{\mathcal{E}})_t| > \lambda\right) \le 2\min\Big\{1, \frac{8}{\pi} \sin(\pi\beta)e^{-\pi\lambda}\Big\}. \tag{4.39}$$

Finally,

$$
\begin{aligned}
(4.40) \qquad \frac{2\arctan 2}{\pi}\sin(\pi\beta)e^{-\pi\lambda} &\le \mathbb{P}\left(\sup_{1\le j\le 2n}\sup_{t>0}|(\mathbf{H}_j * 1_{\mathcal{E}})_t| > \lambda\right)\\
&\le 4n\min\left\{1, \frac{8}{\pi}\sin(\pi\beta)e^{-\pi\lambda}\right\},
\end{aligned}
$$

gives the lower bound in (4.33) with explicit constants. For a single $\mathbf{H}_j$ the upper bound is also independent of dimension. independent.

We summarize the above in the following theorem.

Theorem 4.26. *Suppose d is odd and set $d+1 = 2n$. Let $\mathbf{H}_1, \ldots, \mathbf{H}_d$ be the above $2n \times 2n$ matices so that the operators $T_{\mathbf{H}_1}, \ldots T_{\mathbf{H}_d}$ correspond to the Riez transforms $R_1, \ldots R_d$ as in (4.32). Then for any set $\mathcal{E} \in \mathcal{F}_\infty$ with $0 < \mathbb{P}(\mathcal{E}) < 1$ the distribution of $(\mathbf{H}_j * 1_{\mathcal{E}})$ is given by (4.36) and in particular, it dependents only on the $\beta = \mathbb{P}(\mathcal{E})$. Furthermore, the distribution of $\sup_t |(\mathbf{H}_j * 1_{\mathcal{E}})_t|$ and $\sup_{1\le j\le 2n}\sup_{t>0}|(\mathbf{H}_j * 1_{\mathcal{E}})_t|$ satisfy the exponential bounds given by (4.39) and (4.40), respectively.*

Remark 4.27. *The argument in [47] is based on decomposing the martingale transform as*

$$(\mathbf{R}_j * 1_{\mathcal{E}})_t = Y_t + Z_t,$$

where Y_t is the component parallel to the martingale direction and Z_t is obtained by projecting the integrand K_s in (4.34) onto its orthogonal complement. Then Z_t can be represented as a time–changed Brownian motion B_t run up to the random time $u_0 = \langle Z\rangle_\infty$. It is then shown that u_0 is independent of $\sigma(M_t; t \ge 0)$ and conditioning on u_0,

$$\mathbb{P}\left(\sup_t |(\mathbf{R}_j * 1_{\mathcal{E}})_t| > \lambda \,\middle|\, u_0\right) \ge \frac{1}{2}\mathbb{P}\left(\sup_{0\le u\le u_0}|B_u| > \lambda \,\middle|\, u_0\right).$$

The right-hand side is just the usual Brownian tail with time parameter u_0. Thus the constant γ in (4.33) ultimately depends on the distribution of u_0, and hence on the law of the underlying martingale, not merely on $\mathbb{P}(\mathcal{E})$.

This is in contrast with the Cotlar setting, where the exponential rate equals π and it is independent of $\mathcal{E}$, reflecting the exact identity rather than the decomposition. In particular, for degenerate matrices such as $\mathbf{R}_1$, u_0 may be arbitrarily small, so no exponential rate depending only on $\mathbb{P}(\mathcal{E})$ can be extracted from this argument. See [47] for details.

As it has already being mentioned, in [38, Theorem 1] Davis gave a probabilistic proof of classical result of Stein and Weiss [88, p. 240] that the distribution of the conjugate function of an indicator function of a measurable set A on the circle depends only on its measure. This result follows from the explicit formula given above. We give the proof here for the Hilbert transform on $\mathbb{R}$.

Let $B_t = (X_t, Y_t)$ be Brownian motion in the upper half-space of $\mathbb{R}^2$ starting at the point $(0,a)$, $a > 0$ and let $\tau = \inf\{t : Y_t = 0\}$; its first exit time. Let $A \subset \mathbb{R}$ be Lebesgue measurable with $0 < |A| < \infty$. Its harmonic measure at the point $(0,a)$ is

$$\mathbb{P}_{(0,a)}(B_\tau \in A) = U_{1_A}(0,a) = \frac{1}{\pi}\int_A \frac{a}{|s|^2 + a^2}\, ds = \beta_a.$$

It has the following properties:

$$(1)\ \ (\pi a)\beta_a \to |A| \quad \text{and} \quad (2)\ \beta_a \to 0,\ \text{both as} \ \ a \to \infty.$$

From (4.36) and Remark 4.14 (Itô's formula plus the Cauchy-Riemann equations)

$$\mathbb{P}_{(0,a)}\left(|H1_A(B_\tau) - \beta_a| > \lambda\right) = \frac{2}{\pi}\arctan\left(\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}\right). \tag{4.41}$$

Fix $\lambda > 0$. Since,

$$\begin{aligned}&\pi a \frac{2}{\pi}\arctan\left(\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}\right)\\ &= \frac{2}{\pi}\,\frac{\arctan\left(\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}\right)}{\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}}\ \frac{\sin(\pi\beta_a)}{\beta_a}\ \frac{1}{\sinh(\pi\lambda)}\ \pi a\beta_a,\end{aligned}$$

using (1) and (2), we have, as $a \to \infty$,

$$\frac{\arctan\left(\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}\right)}{\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}} \to 1, \qquad \frac{\sin(\pi\beta_{a)})}{\beta_a} \to \pi.$$

This gives the following

Corollary 4.28.

(4.42)
$$|\{x \in \mathbb{R} : |H1_A(x)| > \lambda\}| = \lim_{a\to\infty} \pi a\,\frac{2}{\pi}\arctan\left(\frac{\sin(\pi\beta_a)}{\sinh(\pi\lambda)}\right) = \frac{2|A|}{\sinh(\pi\lambda)}.$$

From the density formulas restricted to the two boundary components of the strip, for every $\lambda > 0$ we have

$$\mathbb{P}_{(0,a)}\big(B_\tau \in A,\ |H1_A(B_\tau) - \beta_a| > \lambda\big) = \int_{|u|>\lambda} \frac{\sin(\pi\beta_a)}{2(\cosh(\pi u) + \cos(\pi\beta_a))}\, du,$$

and

$$\mathbb{P}_{(0,a)}\big(B_\tau \in A^c,\ |H1_A(B_\tau) - \beta_a| > \lambda\big) = \int_{|u|>\lambda} \frac{\sin(\pi\beta_a)}{2(\cosh(\pi u) - \cos(\pi\beta_a))}\, du.$$

Multiplying by πa and using $\sin(\pi\beta_a) \sim \pi\beta_a$ and $\cos(\pi\beta_a) \to 1$, we obtain

$$\pi a\,\mathbb{P}_{(0,a)}\big(B_\tau \in A,\ |H1_A(B_\tau) - \beta_a| > \lambda\big) \to |A|\int_{|u|>\lambda} \frac{\pi}{2(\cosh(\pi u) + 1)}\, du.$$

By symmetry and the identity $\cosh t + 1 = 2\cosh^2(t/2)$,

$$\begin{aligned}\int_{|u|>\lambda} \frac{\pi}{2(\cosh(\pi u)+1)}\,du &= \pi \int_y^\infty \frac{du}{\cosh(\pi u)+1} \\ &= \frac{\pi}{2}\int_\lambda^\infty \operatorname{sech}^2\Big(\frac{\pi u}{2}\Big)\,du \\ &= \int_{\pi y/2}^\infty \operatorname{sech}^2 v\,dv \\ &= 1-\tanh\Big(\frac{\pi\lambda}{2}\Big) = \frac{2}{e^{\pi y}+1}.\end{aligned}$$

Hence

$$\lim_{a\to\infty} \pi a\,\mathbb{P}_{(0,a)}\big(B_\tau \in A,\ |H1_A(B_\tau)-\beta_a| > y\big) = \frac{2|A|}{e^{\pi y}+1}.$$

On the other hand,

$$\mathbb{P}_{(0,a)}\big(B_\tau \in A,\ |H1_A(B_\tau)-\beta_a| > y\big) = \frac{1}{\pi}\int_{A\cap\{|H1_A(s)-\beta_a|>y\}} \frac{a}{s^2+a^2}\,ds,$$

so

$$\pi a\,\mathbb{P}_{(0,a)}\big(B_\tau \in A,\ |H1_A(B_\tau)-\beta_a| > y\big) = \int_{A\cap\{|H1_A(s)-\beta_a|>y\}} \frac{a^2}{s^2+a^2}\,ds.$$

Letting $a\to\infty$ yields

Corollary 4.29.

$$\big|\{x\in A : |H1_A(x)| > y\}\big| = \frac{2|A|}{e^{\pi y}+1}. \tag{4.43}$$

Similarly,

$$\big|\{x\in A^c : |H1_A(x)| > y\}\big| = \frac{2|A|}{e^{\pi y}-1}. \tag{4.44}$$

Both equalities (4.43), (4.44) were first proved in [67] by different analysis methods.

Versions of the above inequalities hold for the martingale transforms associated with the matrices $\mathbf{H}_j$ in all odd dimensions constructed by the harmonic extension of indicator of Lebesgue sets in $\mathbb{R}^d$. These results do not apply, at least not directly, to the operators $R_j = T_{\mathbf{H}_j}$ since their distributions need not be controlled by those of the corresponding martingale transforms, as is the case for L^p norms, $1\le p<\infty$. For closely related L^p, $LLogL$ and versions of weak-type inequalities that hold in all dimensions and apply to Riesz transforms in other geometric settings, see [18, 76, 78].

These types of arguments have been used to identify the best constants in the Kolmogorov weak-type inequality for the Hilbert transform, as originally done in [38] for $p=1$, and subsequently for the general case of orthogonal martingales in [64] for $1\le p\le 2$. While the weak-type inequalities hold for the Hilbert transform as well as for martingale transforms associated with the matrices $\mathbf{H}_j$, the problem of proving similar estimates for $R_j = T_{\mathbf{H}_j}$ remains open.

In particular, it is not known whether these operators satisfy a weak-type $(1,1)$ inequality with constant independent of the dimension, a problem raised by E. M. Stein [87, Problem (b), p. 203] in his Berkeley 1986 ICM address. It was shown in [65] that the weak-type constant grows at most logarithmically in d, which, to the best of our knowledge, is the best result currently available in the literature. The proof uses a modification of the Calderón-Zygmund decomposition which in its classical form leads to exponential growth with in the dimension.

In [83] a different proof of Janakiraman's result is presented which shows that the problem can be reduced to studying Riesz transforms applied to finite sums of Dirac masses. Even such logarithmic growth does not appear attainable using the current martingale methods.

5. Problem 1: E.M. Stein's inequality for vector of Riesz transforms

5.1. **Brief history.** Set

$$Rf = (R_1 f, R_2 f, \dots, R_d f)$$

and

$$\|R\|_{L^p(\mathbb{R}^d)} = \sup_{\|f\|_{L^p(\mathbb{R}^d)} \le 1} \left\| \left(\sum_{k=1}^{d} |R_k f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}. \tag{5.1}$$

It follows from Calderón-Zygmund theory [84] that

$$\|R\|_{L^p(\mathbb{R}^d)} \le \begin{cases} \frac{C_d}{p-1}, & 1 < p \le 2, \\ C_d(p-1), & 2 \le p < \infty, \end{cases} \tag{5.2}$$

where the constant C_d grows exponentially with d. The behavior in p, as $\to 1$ and $\mathsf{p} \to \infty$, is best possible. In his landmark paper [85] E. M. Stein proved (5.2) with a constant A_p independent of d. His proof uses L^p-inequalities for Littlewood-Paley square functions [86]. His constant, although independent of dimension, did not give the correct behavior in p. It uses the fact that $\|g(f)\|_{L^p(\mathbb{R}^d)}$ is comparable to $\|f\|_{L^p(\mathbb{R}^d)}$, $1 < p < \infty$, with constants depending only on p. Here g denotes the (horizontal, vertical or full) Littlewood-Paley function as in [84, Chapter IV]. While it is well-known that $\|g(f)\|_{L^p(\mathbb{R}^d))} \le b_p \|f\|_{L^p(\mathbb{R}^d)}$ with $b_p \sim O(\sqrt{p})$, as $p \to \infty$ and that this growth is best possible, the best known behavior for the constant a_p in the inequality $\|f\|_{L^p(\mathbb{R}^d)} \le a_p \|g(f)\|_{L^p(\mathbb{R}^d)}$ is $O(p)$, as $p \to \infty$. Hence the Littlewood-Paley function argument will give, at best, $A_p \sim p^{3/2}$, as $p \to \infty$.

It is worth noting here that proving that the behavior of a_p is $O(\sqrt{p})$ for the Littlewood-Paley function g is a well-known open (and perhaps forgotten by now) problem which is related to the sub-gaussian behavior of functions with bounded

Luzin area integral studied by Chang, Wilson and Wolff [35]. For more information on this type of behavior of constants for square functions on $\mathbb{R}^d$, see [16].

A different proof of (5.2) (and a version for Wiener space) was given by P. A. Meyer in [71] using the Burkholder-Gundy martingale square function inequalities to prove the needed Littlewood-Paley inequalities. A similar argument was used in A. Bennett's 1984 thesis [29] written under Stein. These proofs did not yield the correct behavior in p either.

Stein's papers on dimension-free bounds for Riesz transforms, maximal functions, and other classical operators in analysis generated great interest in the harmonic analysis community; interest that continues to this day. In particular, soon after [85] appeared, several alternative proofs of (5.2) were obtained, including probabilistic proofs based on the stochastic integral representation of Gundy–Varopoulos [56], as in [6], as well as the approach in [46] using techniques from the method of rotations. None of these proofs yields the correct behavior in p both as $p \to 1$ and $p \to \infty$.

In [81], G. Pisier gave an elegant proof based on the *transference method* of Coifman and Weiss which extends ideas from the method of rotations to a much broader setting (see, for example, "Some classical examples of transference," [36, pp. 5–9]) together with a projection operator onto the first Wiener chaos. Pisier's argument gives the correct order as $p \to \infty$. However, for $1 < p \leq 2$ his resulting estimate is of order $(p-1)^{-3/2}$, which is not optimal. This is also the same behavior giving in [46].

The correct behavior for the full range of p first appeared in [7], where it is derived from an exponential vector-valued martingale together with a sharp vector-valued good-λ inequality that follows from it. Nevertheless, Pisier's proof remains remarkable: it not only captures the correct growth as $p \to \infty$, but also provides the best known uniform upper bound currently available in the literature for the range $2 \leq p < \infty$.

5.2. **Known upper bounds.** Although there is a vast literature now on sharp inequalities for Riesz transforms and their variants in different geometric settings, the problem of identifying the sharp constant A_p (not just its correct dependance on p) in Stein's (5.2) inequality for any $d > 1$ remains open, except for the trivial case of $p = 2$. Below we present a short survey of some known estimates with the caveat that the list is not exhaustive and that improved or more refined estimates that we are not aware of may be available in the literature. We restrict our discussion to the classical case of Riesz transforms on $\mathbb{R}^d$, which we regard as the fundamental setting and the natural starting point in the search for optimal bound. Thus there are many missing references in our discussion below to similar bounds for Reisz transforms in many different settings with a variety of different techniques such as those in [1, 4, 19, 20, 33, 69, 92], and references therein.

Before we discuss Pisier's theorem, we present other developments related to Stein's inequality on $\mathbb{R}^d$.

Theorem A (T Iwaniec and G. Martin [61]). *Set $\mathbf{H}_{\mathbb{C}} = R_1 + iR_2$ where R_1 and R_2 are the Riesz transforms on $\mathbb{R}^2$. Then,*

$$\|R\|_{L^p(\mathbb{R}^d)} \leq \sqrt{2}\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^d)}, \quad 2 \leq p < \infty. \tag{5.3}$$

Iwaniec and Martin proved this and other related results by extending the classical method of rotations on $\mathbb{R}^d$ to the complex setting on $\mathbb{C}^{2d}$. They called the operator $\mathbf{H}_{\mathbb{C}}$ *"the complex Hilbert transform."* It is also called the *"complex Riesz transform"* in other places in the literature. The value of the norm of its powers, $\|(\mathbf{H}_{\mathbb{C}})^k\|_{L^p(\mathbb{R}^2)}$, $k \in \mathbb{N}$, and is applications, has been investigated by several authors. For some of this literature we refer to [3], [34], [45], [43], [63] and the many references contained therein.

Remark 5.1. *For clarity of comparison with results in the literature (such as [34] and [3]), we remind the reader that we use $\|T\|_{L^p(\mathbb{R}^d)}$ and $\|T\|_{L^p(\mathbb{R}^d;\mathbb{C})}$ to denote the norm of the operator T acting on real-valued and complex-valued functions, respectively. This is an important distinction when dealing with norms of operators that map real-valued functions to complex-valued functions such as $\mathbf{H}_{\mathbb{C}}$ and the Beurling-Ahlfors operators defined in (6.1) below.*

Since the operator $R_1 + iR_2$ on $\mathbb{R}^d$ is a Fourier multiplier extension of $R_1 + iR_2$ on $\mathbb{R}^2$, it follows from de Leeuw, [39], [54, Theorem 2.5.15], that

$$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \leq \|R_1 + iR_2\|_{L^p(\mathbb{R}^d)} \leq \|R\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \tag{5.4}$$

Thus,

$$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \leq \|R\|_{L^p(\mathbb{R}^d)} \leq \sqrt{2}\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)}, \quad 2 \leq p < \infty. \tag{5.5}$$

Clearly a lower bound is given by the norm of either R_1 or R_2. Applying Minkowski and using (2.6) gives

$$\cot\left(\frac{\pi}{2p^*}\right) \leq \|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \leq 2\cot\left(\frac{\pi}{2p^*}\right), \quad 1 < p < \infty \tag{5.6}$$

and hence,

$$\cot\left(\frac{\pi}{2p}\right) \leq \|R\|_{L^p(\mathbb{R}^d)} \leq 2\sqrt{2}\cot\left(\frac{\pi}{2p}\right), \quad 2 \leq p < \infty. \tag{5.7}$$

In [5, Corollary 4.5], the sharp martingale inequalities of Burkholder [32] are used to prove that

$$\|R\|_{L^p(\mathbb{R}^d)} \leq 2(p^*-1) = \begin{cases} \frac{2}{(p-1)}, & 1 < p \leq 2, \\ 2(p-1), & 2 \leq p < \infty. \end{cases} \tag{5.8}$$

This inequality holds for complex-valued functions.

How does this compare with the bound in (5.7) for $p > 2$? Setting $f(p) = (p-1) - \sqrt{2}\cot(\pi/2p)$, it is not difficult to see that there exist a p_0 (≈ 9) such that $f(p_0) = 0$ and $f(p) < 0$ for $2 < p < p_0$ and $f(p) > 0$ for $p > p_0$. Thus there is some small improvement for $2 < p < p_0$. It is clear however that trivially estimating the norm of $\mathbf{H}_{\mathbb{C}}$ from above using Minkowski's inequality is not efficient.

In what follows we aim to provide some better bounds for $\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)}$ and $\|R\|_{L^p(\mathbb{R}^d)}$ for $p \geq 2$. We begin with the classical method or rotations. This method has been used in various palaces in the literature to estimate norm of $(\mathbf{H}_{\mathbb{C}})^k$ when k is odd; see for example [34, 45, 63]. From (2.5), $\mathbf{H}_{\mathbb{C}}$ has Fourier multiplier

$$i\,\frac{\xi_1 + i\xi_2}{|\xi|}$$

and therefore, $(H_{\mathbb{C}})^k$ has multiplier

$$(i)^k e^{ik\theta}, \qquad \xi = (|\xi|\cos\theta, |\xi|\sin\theta).$$

Let

$$H_\phi f(x) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x - te_\phi)}{t}\,dt.$$

be the Hilbert transform H_ϕ in direction $e_\phi = (\cos\phi, \sin\phi)$. Its multiplier is $i\,\text{sgn}(\cos(\theta - \phi))$. Using the Fourier expansion

$$\text{sgn}(\cos t) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}\cos((2m+1)t),$$

we see that only odd angular frequencies occur. Thus, for each odd integer $k \geq 1$,

$$(H_{\mathbb{C}})^k = \omega_k\,\frac{\pi}{2}\,k\int_0^{2\pi} e^{ik\phi} H_\phi\,\frac{d\phi}{2\pi}, \qquad |\omega_k| = 1. \tag{5.9}$$

In particular, it follows immediately

$$\|\,(\mathbf{H}_{\mathbb{C}})^k\,\|_{L^p(\mathbb{R}^2)} \leq k\frac{\pi}{2}\|H\|_p, \;\; 1 < p < \infty, \tag{5.10}$$

which for $k = 1$ improves the bound in (5.7).

For $d = 2$, we obtain from (4.23) and Lemma 4.16 that with

$$\mathbf{R}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{5.11}$$

we can write

$$\mathbf{H}_{\mathbb{C}} = T_{\mathbf{R}_1} + iT_{\mathbf{R}_2},$$

where $T_{\mathbf{R}_1}$ and $T_{\mathbf{R}_2}$ are the projection of the martingale transforms. Applying the vector valued inequality [24, Theorem A] gives a further improvement over (5.10)

$$\|\mathbf{R}_1 + i\mathbf{R}_2\|_{L^p(\mathbb{R}^2)} \leq \sqrt{2}\cot\left(\frac{\pi}{2p}\right), \;\; 2 \leq p < \infty. \tag{5.12}$$

Both, the method of rotations and the martingale method, are not any better than what follows from the elementary inequality $(a^2+b^2)^{p/2} \leq 2^{p/2-1}(a^2+b^2)$ which already improves the Minkowski estimate (5.7) to $\sqrt{2}$. On the other hand, both

(5.10) and (5.12) hold for complex-valued functions in the range of $2 \le p < \infty$. Regardless, we have the following bound

$$\cot\left(\frac{\pi}{2p^*}\right) \le \|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \le \|R\|_{L^p(\mathbb{R}^d)} \le \begin{cases} 2(p^*-1) & 1 < p \le 2, \\ 2\cot\left(\frac{\pi}{2p}\right), & 2 \le p < \infty, \end{cases} \tag{5.13}$$

The problem of determining the exact L^p-norm $\mathbf{H}_{\mathbb{C}}$ (which is the same as the best constant in Stein's inequality for $d = 2$) remains open both in the complex and real case, as far as we know. Thus, further improvements on the upper bound for $\|R\|_{L^p(\mathbb{R}^d)}$ in terms of the norm of $\mathbf{H}_{\mathbb{C}}$ appear unlikely at present. Moreover, the constant $\sqrt{2}$ in the Iwaniec-Martin inequality (5.3), which is obtained in two steps: (1) from the Fourier extension to $\mathbb{C}^n$ combined with de Leeuw's extension theorem and (2) from the complex method of rotations, also seems quite challenging.

Similarly, applying the vector value martingale inequality from [5] on $\mathbb{R}^d$, which gives the constant $\sqrt{2}$ in (5.12) on $\mathbb{R}^2$, would give the dimension depending bound $\|R\|_{L^p(\mathbb{R}^d)} \le \sqrt{d}\cot(\pi/2p)$, for any $2 \le p < \infty$. In addition, the inequality $\|R\|_{L^p(\mathbb{R}^d)} \le 2(p^*-1)$ proved in [5] uses the matrix representation for $\frac{1}{2}R_j$ as in Lemma 4.16. With those matrices, which are no longer orthogonal but have the differential subordination property independent of d, Burkholder's vector-valued inequality gives the bound (p^*-1) and multiplying by 2 gives the bound in (5.8) valid for all $1 < p < \infty$. This it is difficult to see improvements with the techniques discussed above.

5.3. **G. Pisier's bound.** We now state Pisier's theorem inserting the explicit value of C_p given on page 466. For completeness, and because the technique will be discussed further below, we also include its proof. Our presentation follows Pisier's original argument, supplemented with additional clarifications where appropriate in order to make the exposition as accessible as possible and self-contained. The aim of [81], as stated in the introduction, was to give a proof of the inequality for the Ornstein-Uhlenbeck Riesz transforms; Meyer's [71] theorem. Here we only discuss the classical case, [81, Theorem 1.1].

The proof for the Ornstein–Uhlenbeck case [81, Theorem 2.1], although more technical, follows a similar pattern. While the resulting universal bound is not as sharp as in the classical case, it exhibits the same asymptotic behavior as $p \to 1$ and $p \to \infty$. See also [58, Theorem 7.5], where Pisier's proof of Meyer's inequality is adapted to the setting of abstract Wiener space.

Theorem B (G. Pisier 1986 [81])**.** *Let $1 < p < \infty$ and q be its conjugate exponent. For a standard normal random variable X, i.e., $X \sim N(0,1)$, we set $\gamma(p) = \|X\|_p$. Then,*

$$\|R\|_{L^p(\mathbb{R}^d)} \le \begin{cases} \frac{\gamma(q)}{\gamma(p)}\sqrt{\frac{\pi}{2}}\ \|H\|_p, & 1 < p \le 2, \\ \sqrt{\frac{\pi}{2}}\ \|H\|_p, & 2 \le p < \infty. \end{cases} \tag{5.14}$$

Before presenting the proof, we give some upper bounds on $\frac{\gamma(q)}{\gamma(p)}$ for $1 < p \leq 2$. By trivial computation using the normal distribution,

$$\gamma(q) = \left(\frac{2^{q/2}\, \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}} \right)^{1/q}.$$

By Stirling's approximation of the gamma function, $\gamma(q) \sim \sqrt{\frac{q}{e}}, \quad q \to \infty$. On the other hand $\gamma(p) \to \gamma(1) = \sqrt{2/\pi}$ as $p \to 1^+$. Thus

$$\frac{\gamma(q)}{\gamma(p)} \sim \sqrt{\frac{\pi}{2e}} \frac{1}{\sqrt{p-1}}, \quad p \to 1^+.$$

Form this and the fact that $\|H\|_p = \cot\left(\frac{\pi}{2p^*}\right)$, the Big-O behavior, $(1-p)^{-3/2}$, as $p \to 1$, and p as $p \to \infty$ follow as stated in [81, p. 488]. One can go a step further and give the sharp universal bound on $\frac{\gamma(q)}{\gamma(p)}$ for the case $1 < p < 2$. Here we discuss three estimates and give the bound that follows from the theorem in the Corollary 5.3 below.

(1) Nelson–Gross Hypercontractivity [55, 74]**:** Let $(P_t)_{t\geq 0}$ denote the Ornstein–Uhlenbeck (OU) semigroup on $\mathbb{R}$ with the Gaussian measure γ. Then for all $1 < r < s < \infty$, $\|P_t f\|_{L^s(\gamma)} \leq \|f\|_{L^r(\gamma)}$ whenever $s - 1 = (r-1)e^{2t}$. Applying this with $r = 2$, $s - 1 = e^{-2t}$, and $f(x) = x$, which is an eigenfunction of P_t with eigenvalue e^{-t}, gives $\gamma(s) = e^t \|P_t f\|_{L^s} \leq e^t = \sqrt{s-1}$, for $s > 2$. Applying this with $s = q$ and using the fact that $\gamma(1) = \sqrt{2/\pi} \leq \gamma(p)$, since $\gamma(r)$ is increasing by Jensen's inequality, it follows that

$$\frac{\gamma(q)}{\gamma(p)} \leq \sqrt{\frac{\pi}{2}}\sqrt{q-1} = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{p-1}}, \quad 1 < p < 2. \tag{5.15}$$

(2) Gautschi/Kershaw and Stirling A different way to get the upper bound above is as follows; set $q = 2r$ so that

$$\mathbb{E}|X|^{2r} = \frac{2^r \Gamma\left(r + \frac{1}{2}\right)}{\sqrt{\pi}}.$$

From the Gautschi/Kershaw and Stirling inequalities it follows that

$$\Gamma\left(r + \tfrac{1}{2}\right) \leq \sqrt{r}\Gamma(r) \leq \sqrt{r}\sqrt{2\pi}\, r^{r-\frac{1}{2}} e^{-r}, \ \ r \geq 1,$$

This gives,

$$\mathbb{E}|X|^{2r} \leq \frac{2^r\sqrt{r}}{\sqrt{\pi}} \sqrt{2\pi}\, r^{r-\frac{1}{2}} e^{-r} = 2^r\, r^r \sqrt{2}\, e^{-r} \leq 2^r\, r^r, \quad r \geq 1.$$

Taking roots we have $\gamma(q) \leq \sqrt{q}$, for $2 < q < \infty$, equivalent for $1 < p < 2$. This gives the same upper bound as (5.15).

(3) Hypercontractivity and $\log$**-convexity of absolute moments** [75]**:** The Hypercontractivity inequality shows that the function $\varphi(r) = \frac{\gamma(r)}{\sqrt{r-1}}$ is monotone decreasing for $r > 1$ which implies that $\varphi(r) = \frac{\gamma(r)}{\sqrt{r}}$ is decreasing for $r > 2$. From the fact that the absolute moments $M(r) = \mathbb{E}|X|^r$ is log-convex it can be shown that $\varphi(r) = \frac{\gamma(r)}{\sqrt{r}}$ is decreasing for $r > 1$. Thus for any $1 < p < q$, $\frac{\gamma(q)}{\sqrt{q}} \le \frac{\gamma(p)}{\sqrt{p}}$. If $q = p/(p-1)$, then $\sqrt{q/p} = (p-1)^{-1/2}$ and it follows that

$$\frac{\gamma(q)}{\gamma(p)} = \frac{\gamma(q)/\sqrt{q}}{\gamma(p)/\sqrt{p}}\sqrt{\frac{q}{p}} \le \sqrt{\frac{q}{p}} = \frac{1}{\sqrt{p-1}}$$

Remark 5.2. *Although the above observations (all known) may be interesting in the search for sharp constants,* **the real problem** *is to remove the factor* $1/\sqrt{p-1}$ *even at the expense of another constant that does not match the bound for the case* $2 \le p < \infty$.

We summarize the explicit bound in (5.14) as a corollary.

Corollary 5.3.

$$\|R\|_{L^p(\mathbb{R}^d)} \le \begin{cases} \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{p-1}}\ \|H\|_p, & 1 < p \le 2, \\ \sqrt{\frac{\pi}{2}}\ \|H\|_p, & 2 \le p < \infty \end{cases} \tag{5.16}$$

Combining the case $p \ge 2$ with the estaminet in (5.8), we have

Corollary 5.4.

$$\|R\|_{L^p(\mathbb{R}^d)} \le \begin{cases} \frac{2}{p-1}, & 1 < p \le 2, \\ \sqrt{\frac{\pi}{2}}\ \|H\|_p, & 2 \le p < \infty. \end{cases} \tag{5.17}$$

Remark 5.5. *Notice that for* $2 \le p < \infty$ *the constant* $\sqrt{\frac{\pi}{2}} \approx 1.25331413$ *is better than all others discussed above. Of course, it still does not attain the sharp constant* 1 *at* $p = 2$. *It is natural to ask whether the same bound* $\sqrt{\pi/2}\,\|H\|_p$ *might hold for the entire range* $1 < p < \infty$ *in Corollary 5.3. That is, without the additional factor* $(p-1)^{-1/2}$ *appearing for* $1 < p < 2$. *This value is an integral part of the proof and removing it by some simple modification of the argument does not seem possible. Hence a different approach would be required.*

Proof of Theorem B. Consider the product space $(\mathbb{R}^n \times \mathbb{R}^n,\ dx \otimes d\gamma_d)$, where

$$d\gamma_d = (2\pi)^{-d/2} e^{-|y|^2/2}\, dy.$$

For $f \in L^p(\mathbb{R}^d)$, let the function $f(x + ty)$ be define on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. By the translation invariance of the Lebesgue measure,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} |f(x+ty)|^p\, dx\, d\gamma_n = \int_{\mathbb{R}^d} |f(x)|^p\, dx, \tag{5.18}$$

Thus, as Pisier states it, the function $(x, y) \to f(x+ty)$ is an *isometric embedding* $L^p(dx) \hookrightarrow L^p(dx \otimes d\gamma_d)$, for each fixed t.

Define an operator $\mathcal{H}$ acting on functions of (x, y) by

$$(\mathcal{H}f)(x, y) := \frac{1}{\pi}\,\mathsf{p}.v.\int_{\mathbb{R}} \frac{f(x+ty)}{t}\,dt.$$

Then by (5.18),

$$\|\mathcal{H}f\|_{L^p(dx\otimes d\gamma_d)} \le \|H\|_p\,\|f\|_{L^p(dx)}. \tag{5.19}$$

This is the transference argument as in [36].

Pisier now uses the orthogonal projection Q from $L^2(\gamma_d)$ onto the first the first Wiener chaos. For completeness, we elaborate further. Let H_n denote the (probabilist's) Hermite polynomials in one variable,

$$H_n(y) = (-1)^n e^{y^2/2}\frac{d^n}{dy^n}e^{-y^2/2}.$$

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, define the multivariate Hermite polynomial

$$H_\alpha(y) = \prod_{j=1}^{d} H_{\alpha_j}(y_j), \qquad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

The family $\{H_\alpha : \alpha \in \mathbb{N}^d\}$ forms an orthogonal basis for $L^2(\gamma_d)$. The m Wiener chaos is defined as $\mathcal{H}_m = \overline{\text{span}}\{H_\alpha : |\alpha| = m\}$ and the following is the Wiener chaos decomposition on $\mathbb{R}^d$ (see, [58, Theorem 5.1 & Example 5.12])/. That is,

$$L^2(\gamma_d) = \bigoplus_{m=0}^{\infty} \text{span}\{H_\alpha(x) : |\alpha| = m\}. \tag{5.20}$$

In particular, the *first Wiener chaos* is $\mathcal{H}_1 = \text{span}\{H_\alpha : |\alpha| = 1\}$. Since $|\alpha| = 1$ implies $\alpha = e_j$ for some $j = 1, \ldots, d$, and since $H_{e_j}(y) = H_1(y_j) = y_j$ we have $\mathcal{H}_1 = \text{span}\{y_1, \ldots, y_d\}$. Thus the first Wiener chaos consists of all linear functions $y \to a \cdot y$, $a \in R$. For $h \in L^2(\gamma_d)$ define the orthogonal projection onto $\mathcal{H}_1$ by

$$(Qh)(y) = \sum_{j=1}^{d} \langle h, y_j\rangle\, y_j = a \cdot y, \quad a \in \mathbb{R}^d, \tag{5.21}$$

where

$$a_j = \langle h, y_j\rangle = \int_{\mathbb{R}^d} h(y)\, y_j\, d\gamma_d, \quad j = 1, \ldots, d.$$

Since the coordinate functions y_j are i.i.d standard normal random variables, Qh is a normal random variable $X \sim N(0, \sigma^2)$ with $\sigma^2 = \|Qh\|^2_{L^2(\gamma_d)}$. Thus for any $2 \le p < \infty$, the boundedness of Q on $L^2(\gamma_d)$ and Jensen's inequality give that for $2 \le p < \infty$,

$$\begin{aligned} \|Qh\|_{L^p(\gamma_d)} &= \gamma(p)\,\|Qh\|_{L^2(\gamma_d)} \\ &\le \gamma(p)\,\|h\|_{L^2(\gamma_d)} \\ &\le \gamma(p)\,\|h\|_{L^p(\gamma_d)}. \end{aligned} \tag{5.22}$$

By duality, for $1 < 2 \leq p$, this gives

$$\|Qh\|_{L^p(\gamma_d)} \leq \gamma(q)\, \|h\|_{L^p(\gamma_d)}, \tag{5.23}$$

where q is the conjugate exponent of p.

For $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, denote by Q_y the operator Q acting on the y variable. Pisier's crucial identity, his equation (1.5), is:

$$Q_y(\mathcal{H}f)(x,y)) = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{d} y_j\, (R_j f)(x). \tag{5.24}$$

Let us assume this for the moment. Taking the $L^p(dx \otimes d\gamma_d)$ norms of both sides, applying (5.19), (5.22) and (5.23) we have

(5.25)

$$\sqrt{\frac{2}{\pi}} \Big\| \sum_{j=1}^{d} y_j R_j f(x) \Big\|_{L^p(dx\otimes d\gamma_d)} \leq \begin{cases} \gamma(q)\ \|H\|_p\, \|f\|_{L^p(\mathbb{R}^d)}, & 1 \leq p \leq 2, \\ \gamma(p)\ \|H\|_p\, \|f\|_{L^p(\mathbb{R}^d)}, & 2 \leq p < \infty. \end{cases}$$

Exactly as above, for each x, $\sum_{j=1}^{d} y_k R_j f(x)$ is he sum of i.i.d. standard normals and hence a mean 0 normal with variance $\sum_{j=1}^{d} |R_j f(x)|^2$. This gives

$$\Big\| \sum_{j=1}^{d} y_k R_j f(x) \Big\|_{L^p(\gamma_n)} = \gamma(p) \Big(\sum_{j=1}^{n} |R_j f(x)|^2 \Big)^{1/2}, \ \ 1 < p < \infty \tag{5.26}$$

This identity is what we call Gaussian averaging and to which we will return below when we explore asymptotic behavior. Taking the $L^p(dx)$ norm of both sides (5.26) gives

$$\|R\|_{L^p(\mathbb{R}^d)} \leq \begin{cases} \frac{\gamma(q)}{\gamma(p)} \sqrt{\frac{\pi}{2}}\ \|H\|_p, & 1 < p < 2, \\ \sqrt{\frac{\pi}{2}}\ \|H\|_p, & 2 \leq p < \infty, \end{cases} \tag{5.27}$$

which is the statement of the Theorem.

It remains to verify (5.24). As Pisier says, this "is easy to check using the Fourier transform in the x variable." Fix $y \in \mathbb{R}^d$. Applying (2.2) the Fourier transform of the left-hands side of (5.24) in the x variable gives

$$Q_y(\widehat{Hf(\cdot, y)})(\xi) = Q_y\big[i \operatorname{sign}(\xi \cdot y)\big]\, \widehat{f}(\xi) = i\, Q_y\big[\operatorname{sign}(\xi \cdot y)\big]\, \widehat{f}(\xi). \tag{5.28}$$

By definition of the first Wiener chaos,

$$Q_y\big[\operatorname{sgn}(\xi \cdot y)\big] = a \cdot y, \quad a \in \mathbb{R}.$$

Let ρ be orthogonal matrix $\rho \in O(d)$ (the orthogonal group in $\mathbb{R}^d$) which fixes ξ. Since $\operatorname{sign}(\xi \cdot \rho y) = \operatorname{sign}(\xi \cdot y)$ and γ_d is rotationally invariant, we have

$$a \cdot (\rho y) = Q_y\big(\operatorname{sgn}(\xi \cdot \rho y)\big) = Q_y\big(\operatorname{sgn}(\xi \cdot y)\big) = a \cdot y.$$

This shows that a is invariant under all orthogonal transformations fixing ξ and hence it must be a multiple of ξ. This gives $a = \alpha(\xi)\, \xi$ and it follows that

$$Q_y\big(\operatorname{sgn}(\xi \cdot y)\big) = \alpha(\xi) \xi \cdot y, \tag{5.29}$$

To determine $\alpha(\xi)$, set $X = \xi \cdot y \sim N(0, |\xi|^2)$. By orthogonality,

$$\begin{aligned}&\mathbb{E}\Big[\Big(\operatorname{sign}(\xi\cdot y) - Q_y(\operatorname{sign}(\xi\cdot y))\Big)\xi\cdot y\Big]\\ &= \mathbb{E}\Big[\Big(\operatorname{sign}(\xi\cdot y) - \alpha(\xi)\,\xi\cdot y\Big)\xi\cdot y\Big] = 0\end{aligned}$$

Equivalently,

$$\mathbb{E}[\operatorname{sign}(X)X] = \mathbb{E}|X| = \sqrt{\frac{2}{\pi}}|\xi| = \alpha(\xi)\mathbb{E}|X|^2 = \alpha(\xi)|\xi|^2$$

Thus $\alpha(\xi) = \sqrt{\frac{2}{\pi}}\frac{1}{|\xi|}$ and

$$\begin{aligned}Q_y(\widehat{Hf(\cdot,y)})(\xi) &= i\,Q_y\big[\operatorname{sign}(\xi\cdot y)\big]\,\widehat{f}(\xi)\\ &= \sqrt{\frac{2}{\pi}}\,\frac{i\,\xi\cdot y}{|\xi|}\widehat{f}(\xi)\\ &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{d}\frac{i\,\xi_j\,y_j}{|\xi|}\widehat{f}(\xi)\\ &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{d}\widehat{R_jf}(\xi)\,y_j,\end{aligned}$$

which gives (5.24) and completes the proof of the theorem. □

Remark 5.6 (Complex-valued functions; Pisier's method)**.** *As Pisier points out in* [81, Remark, p. 498]*, his method applies to functions taking values in a Banach space with the unconditional martingale differences property, U.M.D. for short. Details on how the constant change are not given. Here we explain the modification for the case of complex-values functions.*

The identity

$$Q_y(\mathcal{H}f)(x,y) = \sqrt{\frac{2}{\pi}}\sum_{j=1}^{d} y_j\,R_jf(x) \tag{5.30}$$

holds for complex-valued functions f, since both $\mathcal{H}$ and Q_y are linear operators. The difference between the real and complex cases appears in the Gaussian moment step. For real coefficients a_j one has the exact identity

$$\|\sum_{j=1}^{d} y_j a_j\|_{L^p(\gamma)} = \gamma(p)\left(\sum_{j=1}^{d} a_j^2\right)^{1/2}.$$

When the coefficients a_j are complex this identity no longer holds. Instead we use the Kahane–Khintchine Gaussian inequality. Let $\alpha(p) = \Gamma\left(\frac{p+2}{2}\right)^{1/p}$. Then

(5.31)

$$\alpha(p)\left(\sum_{j=1}^{d}|a_j|^2\right)^{1/2} \le \left\|\sum_{j=1}^{d} y_j a_j\right\|_{L^p(\gamma)} \le \gamma(p)\left(\sum_{j=1}^{d}|a_j|^2\right)^{1/2}, \; 2 \le p < \infty,$$

(5.32)

$$\gamma(p)\left(\sum_{j=1}^{d}|a_j|^2\right)^{1/2} \le \left\|\sum_{j=1}^{d} y_j a_j\right\|_{L^p(\gamma)} \le \alpha(p)\left(\sum_{j=1}^{d}|a_j|^2\right)^{1/2}, \; 1 < p \le 2$$

With this, (5.27) becomes

$$\|R\|_{L^p(\mathbb{R}^d)} \le \begin{cases} \sqrt{\frac{\pi}{2}}\,\alpha(p)\,\frac{\gamma(q)}{\gamma(p)}, \; \|H\|_p, & 1 < p \le 2, \\ \sqrt{\frac{\pi}{2}}\,\frac{\gamma(p)}{\alpha(p)}\;\|H\|_p, & 2 < p < \infty. \end{cases} \tag{5.33}$$

The constants simplify to

$$\frac{\gamma_p}{\sqrt{2/\pi}\,\alpha_p} = \sqrt{\pi}\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{p+2}{2}\right)}\right)^{1/p} = \sqrt{\pi}\,C_p(2) \le \sqrt{\pi}, \quad 2 \le p < \infty,$$

where the inequality follows from Remark 5.10 below. Similarly,

$$\sqrt{\frac{\pi}{2}}\,\alpha_p\,\frac{\gamma_q}{\gamma_p} \le \sqrt{\frac{\pi}{2}}\,(p-1)^{-1/2}.$$

We summarize the above in the following

Corollary 5.7. *The L^p-norm of the vector of Riesz transforms when acting on complex-valued functions has the bounds*

$$\|R\|_{L^p(\mathbb{R}^d;\mathbb{C})} \le \begin{cases} \sqrt{\frac{\pi}{2}}\,\frac{1}{\sqrt{p-1}}\;\|H\|_p & 1 < p < 2, \\ \sqrt{\pi}\;\|H\|_p, & 2 \le p < \infty. \end{cases} \tag{5.34}$$

In combination with (5.8) which holds for complex-valued functions we have

Corollary 5.8.

$$\|R\|_{L^p(\mathbb{R}^d;\mathbb{C})} \le \begin{cases} \frac{2}{p-1} & 1 < p < 2, \\ \sqrt{\pi}\;\|H\|_p, & 2 \le p < \infty. \end{cases} \tag{5.35}$$

We observe that for complex-valued functions and $d = 2$, the method of rotations (5.9) gives

$$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2;\mathbb{C})} \le \frac{\pi}{2}\cot\left(\frac{\pi}{2p}\right), \; 2 \le p < \infty,$$

and the martingale inequality (5.12) gives

$$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2;\mathbb{C})} \le \sqrt{2}\cot\left(\frac{\pi}{2p}\right),\ \ 2 \le p < \infty,$$

which are better than the estimate in the Corollary 5.8 for the particular case of $d = 2$. The "dimensional dependence" of the constant $\sqrt{2}$ in the second inequality will be clarified below.

5.4. **Asymptotic bounds for vector of Riesz transforms, averaging.** A standard technique in probability and analysis when estimating norms of operators in Hilbert spaces is to average with respect to Rademacher, Gaussian, or uniform distributions. An example of this is Pisier's (5.26) where the length of the vector $v = Rf(x) = (R_1 f(x), \ldots, R_d f(x))$ is written as the expectation of a mean 0 variance $|v|^2$ normal random variable. The crux of Pisier's proof is to combine this simple tool with the deeper idea of projecting onto the first Wiener chaos, thereby reducing the dimensional dependence from $\mathbb{R}^d$ to $\mathbb{R}$. In what follows we will explore averaging techniques to obtain sharper asymptotic behavior of L^p-norms as $p \to \infty$. We recall the following

Conjecture (Problem 6, p. 819 [8])

$$\|R\|_{L^p(\mathbb{R}^d)} = \|H\|_p = \cot\left(\frac{\pi}{2p^*}\right). \tag{5.36}$$

Although this may be false, we have the the following

Theorem 5.9.

$$\lim_{p\to\infty} \frac{\|R\|_{L^p(\mathbb{R}^d)}}{\|H\|_p} = 1. \tag{5.37}$$

Proof. To proof this, we will use the averaging technique. Given Theorem B which produces such a good uniform bound one might expect that applying the averaging technique with Gaussians would yield the best asymptotic result of the three methods mentioned above. However, as it turns out, averaging with the Gaussian and uniform distributions give the same result while averaging with the Rademacher distribution is less effective. We explore all three averaging distributions.

We begin with uniform averaging. We denote by $d\sigma$ the normalized surface measure on the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$. For $d = 2$ this is just the arc-length measure on the circle $\mathbb{S}^1$. That is, $d\sigma(\theta) = \frac{d\theta}{2\pi}$ on $[0, 2\pi)$. Recall that (by rotational invariance of $d\sigma$) for any $x \in \mathbb{R}^d$, $0 < p < \infty$,

$$\int_{\mathbb{S}^{d-1}} |\omega \cdot x|^p \, d\sigma(\omega) = |x|^p \int_{\mathbb{S}^{d-1}} |\omega_1|^p \, d\sigma(\omega).$$

Define

$$C_p(d) = \|\omega_1\|_p = \left(\int_{\mathbb{S}^{d-1}} |\omega_1|^p \, d\sigma(\omega)\right)^{1/p} = \left(\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d+p}{2}\right)}\right)^{1/p}. \tag{5.38}$$

Remark 5.10. *Fix d. Then $C_p(d)$, as a function of p, has the following simple properties that will be use several times below:*

(i) $C_2(d) = \frac{1}{\sqrt{d}}$. *This follows from the fact that* $\Gamma\left(\frac{d+2}{2}\right) = \frac{d}{2}\Gamma(\frac{d}{2})$, $\Gamma(3/2) = \sqrt{\pi}/2$.

(ii) $C_p(d)$ *is increasing, equivalently* $\frac{1}{C_p(d)}$ *is decreasing, in* p. *This follows from Jensen's inequality.*

(iii) $C_p(d) \to \|\omega\|_\infty = 1$, *as* $p \to \infty$. *This is a simple exercise in introductory analysis or by looking at asymptotic behavior of the* Γ *function.*

For $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{S}^{d-1}$ define the directional Riesz transform

$$R_\omega f := \omega_1 R_1 f + \cdots + \omega_d R_d f.$$

Then,

$$\left(|R_1 f(x))|^2 + \cdots + |R_d f(x)|^2\right)^{p/2} = \frac{1}{(C_p(d))^p} \int_{\mathbb{S}^{d-1}} |R_\omega f(x)|^p d\sigma(\omega).$$

Integrating on x gives

$$\|Rf\|^p_{L^p(\mathbb{R}^d)} = \frac{1}{(C_p(d))^p} \int_{\mathbb{S}^{d-1}} \|R_\omega f(x)\|^p_{L^p(\mathbb{R}^d)} d\sigma(\omega).$$

We claim that for every $\omega \in \mathbb{S}^{d-1}$ and $1 < p < \infty$,

$$\|R_\omega\|_{L^p(\mathbb{R}^d)} = \|R_1\|_{L^p(\mathbb{R}^d)}.$$

This is proved exactly as in Remark 4.21. Let $U \in SO(d)$ be an orthogonal matrix with $Ue_1 = \omega$. Define $(Uf)(x) := f(U^{-1}x)$. Then U is an L^p-isometry. A Fourier multiplier computation shows $R_\omega = UR_1U^{-1}$. Indeed,

$$\widehat{R_\omega f}(\xi) = i\,\frac{\omega \cdot \xi}{|\xi|}\,\widehat{f}(\xi) = i\,\frac{e_1 \cdot (U^{-1}\xi)}{|U^{-1}\xi|}\,\widehat{f}(\xi),$$

and similarly,

$$U\widehat{R_1 U^{-1}} f(\xi) = R_1\widehat{(U^{-1}} f)(U^{-1}\xi) = i\,\frac{e_1 \cdot (U^{-1}\xi)}{|U^{-1}\xi|}\,\widehat{f}(\xi).$$

Thus we have

$$\|Rf\|_{L^p(\mathbb{R}^d)} \le \frac{1}{C_p(d)} \|R_1\|_{L^p(\mathbb{R}^d)} = \frac{1}{C_p(d)} \|H\|_p, \quad 1 < p < \infty. \tag{5.39}$$

Combined with the trivial bound from below we have

$$1 \le \frac{\|R\|_{L^p(\mathbb{R}^d)}}{\|H\|_p} \le \frac{1}{C_p(d)},$$

and (5.37) follows from (iii) in Remark 5.10, completing the proof of the proposition. □

For fix d, (5.39) can be used to give universal bounds for $\|R\|_{L^p(\mathbb{R}^d)}$ which, unfortunately, are not dimension independent and in fact they are of order $\sqrt{d}$. More precisely, since $\frac{1}{C_p(d)}$ is decreasing in p we have

$$\|R\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{C_1(d)}\|H\|_p, \;\; 1 < p < \infty \tag{5.40}$$

or the explicit bound bound using (i)

$$\|R\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{C_2(d)}\|H\|_p = \sqrt{d}\,\|H\|_p, \;\; 2 \leq p < \infty \tag{5.41}$$

These type of bounds can just be obtained from ℓ^2, ℓ^1 comparisons. But it is interesting to note that the bound for $p \geq 2$ in the preceding inequity is exactly the same bound that follows from the vector-valued version of the orthogonal subordination argument using [24, Theorem A]. It is precisely for this reason that Burkholder's inequality is used there to obtain the bound $2(p^* - 1)$ in (5.8). More precisely consider the matrices $\mathbf{R}_j$ with the orthogonality property for which the projection operator $T_{\mathbf{R}_j} = R_j$, and the matrices $\mathbf{R}_j^1$ which do not have the orthogonal property but for which $T_{\mathbf{R}_j^1} = \frac{1}{2}R_j$, as in Lemma 4.16. For a real-valued martingale M_t, the quadratic variation of the vector valued martingale transform has $\sum_{j=1}^d \langle \mathbf{R}_j * M_t\rangle \leq d\langle M_t\rangle$ and the inequality from [24, Theorem A] (also [5]) gives (5.41). On the other hand, $\sum_{j=1}^d \langle \mathbf{R}_j^1 * M_t\rangle \leq \langle M_t\rangle$ and Burkholder's vector-valued inequality, which does not require orthogonally give, the bound $(p^* - 1)$ for the vector of $1/2$ Riesz transforms. This is the argument that proves (5.8).

Next we consider Gaussian averaging. This, essentially, follows from (5.26). Here we re-do the identity with different notation to simplify matters a bit. Set $Z = (Z_1, \dots, Z_d)$, where Z_j are i.i.d. $N(0,1)$ random variables. Then $|Z|^2$ is a chi-square random variable with d-degrees of freedom, that is, a Gamma($d/2$, 2) distribution. From this it follows that the density of $|Z|$ is given by

$$f_{|Z|}(r) = \frac{1}{2^{\frac{d}{2}-1}\Gamma\big(\frac{d}{2}\big)}\, r^{d-1}\, e^{-r^2/2}, \quad r \geq 0$$

and

$$\mathbb{E}|Z|^p = 2^{p/2}\frac{\Gamma(\frac{p+d}{2})}{\Gamma(\frac{d}{2})}, \quad 0 < p < \infty.$$

Setting

$$m(p) = \mathbb{E}|N(0,1)|^p = \frac{2^{p/2}\,\Gamma\Big(\frac{p+1}{2}\Big)}{\sqrt{\pi}}$$

we see that if $v(x) = (R_1 f(x), \dots, R_d f(x))$, then $Z \cdot v \sim N(0, |v|^2)$. Hence

$$\big(|R_1 f(x)|^2 + \cdots + |R_d f(x)|^2\big)^{p/2} = \frac{1}{m_p}\,\mathbb{E}\big|Z_1 R_1 f(x) + \cdots + Z_d R_2 f(x)\big|^p, \tag{5.42}$$

for all $1 < p < \infty$, which is the same as identity (5.26). Integrating both sides in x and using Fubini's theorem

(5.43) $$\|Rf\|_{L^p(\mathbb{R}^d)}^p = \frac{1}{m_p}\,\mathbb{E}\big\|\, Z_1R_1f + \cdots + Z_dR_2f\,\big\|_{L^p(\mathbb{R}^d)}^p.$$

Fix $\omega \in \Omega$ and let $z = Z(\omega) \in \mathbb{R}^d$, then (by taking Fourier transform)

$$z_1R_1 + \cdots + z_dR_d = |z|\,R_\psi, \qquad \psi = \frac{z}{|z|} \in \mathbb{S}^{d-1}.$$

As before, the rotation invariance of the Riesz transforms gives

$$\|R_\psi\|_{L^p(\mathbb{R}^d)} = \|R_1\|_{L^p(\mathbb{R}^d)}.$$

Therefore,

$$\|(z_1R_1 + \cdots + z_dR_d)f\|_{L^p(\mathbb{R}^d)}^p \le |z|^p\,\|R_1\|_{L^p(\mathbb{R}^d)}^p\,\|f\|_{L^p(\mathbb{R}^d)}$$

From this and (5.43),

(5.44) $$\|R\|_{L^p(\mathbb{R}^d)} \le \left(\frac{\mathbb{E}|Z|^p}{m_p}\right)^{1/p}\|R_1\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty.$$

Hence,

$$\left(\frac{\mathbb{E}|Z|^p}{m_p}\right)^{1/p} = \left(\frac{\sqrt{\pi}\,\Gamma(\frac{p+d}{2})}{\Gamma(\frac{d}{2})\Gamma\left(\frac{p+1}{2}\right)}\right)^{1/p} = \frac{1}{C_p(d)},$$

which is the same as the constant in (5.38). Thus the Gaussian averaging gives the same constant as uniform averaging on the sphere and also proves (5.37)

Finally, let $(\varepsilon_1, \ldots, \varepsilon_d)$ be the i.i,d Rademacher random variables $\mathbb{P}\,(\varepsilon_j = \pm 1) = \frac{1}{2}$. Recall Khintchine inequality with the best constant:

$$\left(\sum_{j=1}^d a_j^2\right)^{1/2} \le K_p\left(\mathbb{E}\Big|\sum_{j=1}^d a_j\varepsilon_j\Big|^p\right)^{1/p},$$

$$K_p = \begin{cases} \left(\frac{\sqrt{\pi}}{2^{p/2}\,\Gamma(\frac{p+1}{2})}\right)^{1/p}, & 1 < p \le 2, \\ 1, & 2 \le p < \infty. \end{cases}$$

With this and the same arguments as above, we obtain

(5.45) $$\|R\|_{L^p(\mathbb{R}^d)} \le \sqrt{d}\;K_p\|R_1\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty.$$

Thus Rademacher averaging does not improve asymptotic bounds as Gaussian and uniform averaging do.

Let us consider the case $d = 2$. Checking that $C_1(2) = \frac{2}{\pi}$, inequalities (5.40), (5.41) reduce to

(5.46) $$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \le \frac{\pi}{2}\|H\|_p, \;\; 1 < p < \infty,$$

and

(5.47) $$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \leq \sqrt{2}\,\|H\|_p, \;\; 2 \leq p < \infty,$$

respectively. The first is inequality (5.10) obtained by the method of rotations and the second is inequality (5.12) obtained by the martingale transforms. Both trivial form the elementary inequality as already mentioned and not as sharp as Pisier's which gives:

(5.48) $$\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)} \leq \sqrt{\frac{\pi}{2}}\,\|H\|_p, \;\; 2 \leq p < \infty,$$

In addition, (5.37) becomes

(5.49) $$\lim_{p\to\infty} \frac{\|\mathbf{H}_{\mathbb{C}}\|_{L^p(\mathbb{R}^2)}}{\|H\|_p} = 1.$$

In [34, Conjecture 1.5, p. 3], it is conjectured that for all $k \in \mathbb{N}$ and $p \geq 2$,

$$\|(\mathbf{H}_{\mathbb{C}})^k\|_{L^p(\mathbb{R}^2;\mathbb{C})} = \frac{\Gamma(1/p)\,\Gamma(1/q + k/2)}{\Gamma(1/q)\,\Gamma(1/p + k/2)},$$

where $q = \frac{p}{p-1}$ and that when $1 < p < 2$ the identity should still hold, but with the roles of p and q reversed.

Taking $k = 1$ and setting

$$D(p) := \frac{\Gamma(1/p)\,\Gamma(1/q + 1/2)}{\Gamma(1/q)\,\Gamma(1/p + 1/2)},$$

one can check that

(5.50) $$\lim_{p\to\infty} \frac{D(p)}{\|H\|_p} = \frac{\pi}{4}.$$

6. Problem 2: T. Iwaniec's conjecture

6.1. **Brief history.** The Beurling-Ahlfors transform is the singular integral operator acting on complex-valued functions on $L^p(\mathbb{C})$, i.e., $f : \mathbb{C} \to \mathbb{C}$, defined by

(6.1) $$\mathcal{B}f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\, dm(w),$$

where dm is the Lebesgue measure on $\mathbb{C}$ identified as $\mathbb{R}^2$. As a Fourier multiplier,

$$\widehat{\mathcal{B}f}(\xi) = \frac{\overline{\xi}}{\xi}\widehat{f}(\xi) = \frac{\overline{\xi}^2}{|\xi|^2}\widehat{f}(\xi)$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, it follows that

(6.2) $$(\mathbf{H}_{\mathbb{C}})^2 = (R_1 + iR_2)^2 = -(R_2^2 - R_1^2 - 2iR_1R_2) = -\mathcal{B}$$

Thus he operator appearing in the Iwaniec-Martin inequality (5.3) is the square root of $-\mathcal{B}$.

The Beurling-Ahlfors transform has been extensively studied in the literature in large part due to its connections to many areas of analysis such as regularity theory for quasiconformal mappings, partial differential equations and the well-known 1982 conjecture of T. Iwaniec [60] concerning its L^p-norm. Using explicit computations with (nearly) extremal functions, it was proved by O. Lehto (see for example [68]) that $(p^* - 1) \leq \|\mathcal{B}\|_{L^p(\mathbb{C};\mathbb{C})}$ and Iwaniec's conjecture asserts that $\|\mathcal{B}\|_{L^p(\mathbb{C};\mathbb{C})} = (p^* - 1)$. As before, the case $p = 2$ is trivial via the Fourier transform. For a sample on the large literature related to this conjecture which remains open for all $p \neq 2$, see [2, 8, 12, 17, 42, 61, 68, 89, 91], and the many references contained therein. To the best of our knowledge the following is the best known upper bound valid for all $1 < p < \infty$:

$$\|\mathcal{B}\|_{L^p(\mathbb{C};\mathbb{C})} \leq 1.575(p^* - 1). \tag{6.3}$$

This bound is proved in [12].

The first explicit bound $\|\mathcal{B}\|_{L^p(\mathbb{C};\mathbb{C})} \leq 4(p^* - 1)$ was proved in [5] using the representation in Lemma 4.16 for second order Riesz transforms as projections of martingale transforms and applying the $(p^* - 1)$ bound for martingale transforms of Burkholder [32]; (1) in Theorem A. The bound was improved to $2(p^* - 1)$ in [73] using Bellman function techniques for the heat equation. In [15] the martingale arguments in [5] are redone to represent the second order Riesz transforms as martingale transforms of space-time Brownian martingales. This approach, which subsequently has seen many uses in different geometry settings beyond Euclidean spaces (see for example [11] and the many references contained therein) reproves the $2(p^* - 1)$ for $\mathcal{B}$ and gives the bound $(p^* - 1)$ for the operators $R_j^2 - R_k^2$ and $2R_jR_k$ for any $j \neq k$ for any $d \geq 2$. These bounds were shown to be sharp in [49]. There are also estimates improving the bounds for $\|\mathcal{B}\|_p$ asymptotically as $\mathsf{p} \to \infty$. See for example [44]. We return to the asymptotics below.

6.2. **Conformal pairs.**

Definition 6.1. *Let A and B be two real-valued $d \times d$ matrices. We will say that the pair (A, B) is $d \times d$ conformal if it satisfies the following two properties:*

$$(i) \quad |Ax| = |Bx| \quad \text{and} \quad (ii) \quad \langle Ax, Bx\rangle = 0, \quad \text{for all } x \in \mathbb{R}^d. \tag{6.4}$$

Remark 6.2. *We arrived at this definition based on the example of interest given in (6.15) below. The "conformal pairs" considered here are closely related to classical constructions in matrix analysis. The conditions $A^TA = B^TB$, and that A^TB is skew-symmetric, imply that A and B share the same positive semidefinite factor in their polar decompositions, and that B is obtained from A by composing with an orthogonal complex structure on the image of A. This is also related to the notion of partial isometries of operators on Hilbert spaces; see Horn and Johnson's "Matrix Analysis" [57].*

Definition 6.3 (Martingale transform norm)**.** *Let A be a real $m \times n$ matrix. For $1 < p < \infty$, define the operator norm of the martingale transform induced by A*

on $L^p(\Omega)$ by

$$\|A\|_{L^p(\Omega)} := \sup_{K\not\equiv 0} \frac{\|A * M\|_p}{\|M\|_p}, \qquad M_t = \int_0^t K_s \cdot dB_s, \tag{6.5}$$

where B_s is n–dimensional Brownian motion and K ranges over predictable $\mathbb{R}^n$–valued processes with $\int_0^t |K_s|^2\, ds < \infty$, a.s.

For a A real $m \times n$ matrix, Burkholder's inequality (3.2) gives

$$\|A\|_{L^p(\Omega)} \le (p^* - 1)\, \|A\|_{\ell^2\to\ell^2}, \qquad 1 < p < \infty, \tag{6.6}$$

where $\|A\|_{\ell^2\to\ell^2} = \sup_{|x|=1} |Ax|$ is the Euclidean operator norm.

Lemma 6.4 (Rotational invariance for conformal martingales)**.** *Let $Z_t = X_t + iY_t$ be a conformal martingale on the Brownian filtration. Then for every $\theta \in \mathbb{R}$ and every $t \ge 0$,*

$$X_t \cos\theta + Y_t \sin\theta \overset{d}{=} X_t,$$

and consequently, for $1 < p < \infty$,

$$\|X \cos\theta + Y \sin\theta\|_p = \|X\|_p.$$

Proof. Set $\sigma_t := \langle X \rangle_t = \langle Y \rangle_t$. By the Dambis–Dubins–Schwarz theorem [26, p.172], there exists a standard planar Brownian motion W such that $(X_t, Y_t) = (W^1_{\sigma_t}, W^2_{\sigma_t})$. Let $R_\theta \in SO(2)$ be the rotation matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Then

$$(X_t \cos\theta + Y_t \sin\theta,\ -X_t \sin\theta + Y_t \cos\theta) = R_\theta(X_t, Y_t) = R_\theta W_{\sigma_t}.$$

Since $R_\theta W$ is again standard planar Brownian motion, $R_\theta W_{\sigma_t}$ has the same law as W_{σ_t}, hence $X_t \cos\theta + Y_t \sin\theta \overset{d}{=} X_t$. Taking L^p norms proves the lemma. □

Theorem 6.5 (Conformal martingales transforms; real-valued)**.** *Let $M_t = \int_0^t K_s \cdot dB_s$ be a martingale on the filtration of d-dimensional Brownian motion as in (3.1). Let (A, B) be a conformal $d\times d$ pair and suppose in addition that $\|A\| = \|B\| \le 1$. Define $X_t := A * M_t$ and $Y_t := B * M_t$. Set*

$$Z_t := X_t + iY_t = (A + iB) * M_t.$$

Then Z_t is a conformal martingale and

$$\|A + iB\|_{L^p(\Omega)} \le \frac{1}{C_p(2)}(p^* - 1), \quad 1 < p < \infty, \tag{6.7}$$

where $C_p(2)$ is as in Remark 5.10.

Proof. The fact that Z_t is conformal follows trivially from the fact that (A, B) is a $d \times d$ conformal pair. From Lemma 6.4 and the averaging property we have

$$\mathbb{E}|Z_t|^p = \mathbb{E}\left(|X_t|^2 + |Y_t|^2\right)^{p/2} = \frac{1}{C_p(2)^p}\int_{\mathbb{S}^1} \mathbb{E}|X\cos\theta + Y\sin\theta|^p d\sigma$$
$$= \frac{1}{C_p(2)^p}\|X\|_p^p, \quad 1 < p < \infty$$

Thus,

$$\|(A+iB)*M_t\|_p \leq \frac{1}{C_p(2)}\|(A+iB)*M_t\|_p \tag{6.8}$$
$$\leq \frac{1}{C_p(2)}(p^*-1)\|M_t\|_p, \quad 1 < p < \infty,$$

where we used Burkholder inequality for the last step. □

Theorem 6.6 (Conformal martingale transforms; complex-valued)**.** *Let $M_t = \int_0^t H_s \cdot dB_s$ and $N_t = \int_0^t K_s \cdot dB_s$ be two martingales as in (3.1) and (A, B) conformal $d \times d$ pair with $\|A\| = \|B\| \leq 1$. Define*

$$Z_t = X_t + iY_t = (A+iB)*(M_t + iN_t),$$

where

$$X_t = (A*M_t - B*N_t), \qquad Y_t = (A*N_t + B*M_t).$$

Then Z_t is a conformal martingale and

$$\|(A+iB)*(M_t+iN_t)\|_p \leq \frac{\sqrt{2}}{C_p(2)}(p^*-1)\|M_t+iN_t\|_p, \quad 1 < p < \infty. \tag{6.9}$$

Proof.

$$\langle X\rangle_t = \int_0^t |AH_s - BK_s|^2\,ds, \qquad \langle Y\rangle_t = \int_0^t |AK_s + BH_s|^2\,ds,$$

and

$$\langle X, Y\rangle_t = \int_0^t (AH_s - BK_s)\cdot(AK_s + BH_s)\,ds.$$

Expanding the norms we obtain

$$|AH_s - BK_s|^2 = |AH_s|^2 + |BK_s|^2 - 2(AH_s)\cdot(BK_s),$$
$$|AK_s + BH_s|^2 = |AK_s|^2 + |BH_s|^2 + 2(AK_s)\cdot(BH_s).$$

This together with (i) and (ii) shows that Z_t is a conformal.

By Lemma 6.4 we have

$$\|Z_t\|_p \leq \frac{1}{C_p(2)}\|X_t\|_p, \qquad 1 < p < \infty,$$

which in this case is equivalent to

$$\|(A+iB)*(M_t+iN_t)\|_p \leq \frac{1}{C_p(2)}\|A*M_t - B*N_t\|_p, \qquad 1 < p < \infty. \tag{6.10}$$

For the pair (A, B) $d \times d$ matrices define the $d \times 2d$ block matrix

$$T = [A \;\; B] = \begin{pmatrix} a_{11} & \cdots & a_{1d} & b_{11} & \cdots & b_{1d} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{d1} & \cdots & a_{dd} & b_{d1} & \cdots & b_{dd} \end{pmatrix}$$

so that $X_t = A * M_t - B * N_t = T * \mathbb{M}_t$ where,

$$\mathbb{M}_t := \begin{pmatrix} M_t \\ -N_t \end{pmatrix}.$$

Since $\|A\| = \|B\| \leq 1$, $\|T\| \leq \sqrt{2}$, (6.10) combined with Burkholder's inequality give

$$\|(A + iB) * (M_t + iN_t)\|_p \leq \frac{\sqrt{2}}{C_p(2)}(p^* - 1)\|M_t + iN_t\|_p, \quad 1 < p < \infty.$$

□

As we have already seen in Section 4.1, conformal martingales and their connections to holomorphic functions on $\mathbb{C}$ and $\mathbb{C}^n$ have been studied for many years. For many example on $\mathbb{C}$ we refer the reader to [47]. The connections of conformal martingales to singular integral, beyond the Hilbert transforms, first emerged with

Problem 6.7. [5, p. 599] *Determine the best constant C_p in the inequality.*

$$\|Y\|_p \leq C_p \|X\|_p, \tag{6.11}$$

when Y is a conformal martingale subordinate to X.

The inequality implies bounds for the p-norm of the Beurling-Ahlfors operator. While Burkholder's inequality already yields the bound $(p^* - 1)$, one expects the conformal condition to lead to a smaller constant. This problem was the impetus for the investigations in [12], where a slight modification of Burkholder's arguments led to the following result

Theorem A. *([12]) Suppose that Y and X are $\mathbb{R}^2$-valued martingales, with Y conformal and subordinate to X, and X arbitrary. Then*

$$\|Y\|_p \leq \sqrt{\frac{p(p-1)}{2}}\|X\|_p, \quad 2 \leq p < \infty. \tag{6.12}$$

Applying this to the the martingales in Theorem 6.5 (real case) and Theorem 6.6 (complex case) give, respectively,

$$\|(A + iB)M\|_p \leq \sqrt{p(p-1)}\, \|M\|_p, \quad 2 \leq p < \infty, \tag{6.13}$$

and

$$\|(A + iB) * (M + iN)\|_p \leq \sqrt{2p(p-1)}\, \|M + iN\|_p, \quad 2 < p < \infty. \tag{6.14}$$

These bounds, applied to conformal pairs (A_0, B_0) in (6.16) together with an interpolation argument using the fact that $\|\mathcal{B}\|_2 = 1$, yield the bound (6.3) proved in [12]. Here we apply the same method to general pairs of $d \times d$ conformal matrices to obtain the corresponding result on $\mathbb{R}^d$.

As mentioned in Remark 6.2, our definition of conformal pairs (A, B) was motivated by the following example.

Example 6.8. *Fix j, $k \in \{1, \ldots, d\}$ with $j \neq k$. Let $\mathcal{A}^{rs}$ denote the $d \times d$ matrix whose only nonzero entry is $(\mathcal{A}^{rs})_{r,s} = -1$. Define the diagonal and off–diagonal matrices, by*

$$A^{(j,k)} := \mathcal{A}^{kk} - \mathcal{A}^{jj}, \quad B^{(j,k)} = \mathcal{A}^{jk} + \mathcal{A}^{kj}. \tag{6.15}$$

Then for every $x \in \mathbb{R}^d$,

$$(A^{(j,k)}x)_j = x_j, \quad (A^{(j,k)}x)_k = -x_k, \qquad (B^{(j,k)}x)_j = -x_k, \quad (B^{(j,k)}x)_k = -x_j,$$

and all other components vanish. Consequently,

$$|A^{(j,k)}x|^2 = x_j^2 + x_k^2 = |B^{(j,k)}x|^2, \quad A^{(j,k)}x \cdot B^{(j,k)}x = 0.$$

Thus the pair $(A^{(j,k)}, B^{(j,k)})$ satisfies the conformal conditions (i), (ii) in (6.4) and moreover $\|A^{(j,k)}\| = \|B^{(j,k)}\| = 1$. Hence, Theorems 6.5 and 6.6 apply along with the estimates (6.13) and (6.14).

Remark 6.9. *The above matrices are $d \times d$ projections of the $(d+1) \times (d+1)$ matrices define in (4.23)-(4.27).*

Of particular interest here is the case when $d = 2$. Define the 2×2 matrices A_0, B_0 and their sum $A_0 + iB_0$ by

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \; A_0 + iB_0 = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}. \tag{6.16}$$

Then (A_0, B_0) is a conformal pair and $A_0 + iB_0$ is the matrix in [15] that gives the Beurling-Ahlfors operator as a conditional expectation of a martingale transform.

6.3. **Projections of conformal martingales.** As we saw in Lemma 4.16, the second-order Riesz transforms can be written as conditional expectations of martingale transforms, as was originally shown by Gundy and Varopoulos in [56], using the harmonic extension $U_f(x, y) = P_y f(x)$, where P_y is the Poisson semigroup (convolution with the Poisson kernel) composed with Brownian motion (B_t, Y_t) in the upper half-space $\mathbb{R}^{d+1}_+ = \{x \in \mathbb{R}^d, y > 0\}$. However, the alternative representation based on the heat extension $V_f(x, t) = T_t f(x)$ where T_t is the heat semigroup (convolution with the heat kernel) and the underlying process is space–time Brownian motion $(B_t, a - s)$, $0 < s \leq a$, improves their L^p-norm estimates by a factor of 2. This simple and natural modification of the Gundy-Varopoulos representation first appeared in [15].

Fix $a > 0$ and let $S_t = (W_t, a - t)$, $0 < s \leq a$, denote space-time Brownian motion started at $(0, a)$ in $\mathbb{R}^d \times (0, \infty)$ where W_t is standard Brownian motion on the $\mathbb{R}^d$. Let $\mathbb{P}_x$ and $\mathbb{E}_x$ denote the probability and expectation for processes starting at the point (x,a) and $\mathbb{P}^a$ denote the "probability" measure associated with the process with initial distribution $m \otimes \delta_a$ and by $\mathbb{E}^a$ the corresponding expectation.

Let $F = f_1 + if_2 \in C_c^\infty(\mathbb{R}^d)$ and let $V_{f_1}(x,t) = T_t f_1(x)$, and similarly for f_2, be their heat extensions to the upper half-space. We denote by ∇V_f the gradient of V in the x variable. By Itô's formula that for $0 < t \le a$,

$$\begin{aligned} V_F(S_t) - V_F(S_0) &= \int_0^t \nabla V_F(S_t)\cdot dW_t \\ &= \int_0^t \nabla V_{f_1}(S_t)\cdot dW_t + i\int_0^t \nabla V_{f_2}(S_t)\cdot dW_t, \\ &= M_t + iN_t \end{aligned}$$

hence M and N are Brownian martingales of the form (3.1). Let $A^{(j,k)}$ and $B^{(j,k)}$ be as in (6.15).

We define the projection operator

$$\begin{aligned} \mathcal{T}^a_{(A^{(j,k)},B^{(j,k)})}F(x) &= \mathbb{E}^a\Big[(A^{jk}+iB^{jk})*[M_t+iN_t]\Big|S_a=(x,0)\Big] \\ &= \mathbb{E}^a\Big[(A^{(j,k)}+iB^{(j,k)})*[M_t+iN_t]\Big|W_a=x\Big]. \end{aligned} \tag{6.17}$$

By Proposition 2.2 in [15],

(6.18)

$$\lim_{a\to\infty}\mathcal{T}^a_{(A^{(j,k)},B^{(j,k)})}F(x) = \mathbb{B}_{(A^{(j,k)},B^{(j,k)})}F = (R^{(jj)} - R^{(kk)} + 2iR^{(jk)})F, \text{ in } L^2,$$

where R^{jk} are the second order Riesz transforms

$$R^{(jk)}f(x) = \int_0^\infty \frac{\partial^2 V_f(x,t)}{\partial x_j \partial x_k}\,dt = \frac{\partial^2}{\partial x_j \partial x_k}(-\Delta)^{-1}f(x), \tag{6.19}$$

$$\widehat{R^{(jk)}f}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}\widehat{f}(\xi).$$

Before we proceed to discuss L^p boundedness properties, let us look a bit more carefully at the martingale structure of these operators. Recall that $j,k \in \{1,\ldots,d\}$ and $j \neq k$. To simplify notation, set

$$\mathcal{B}^{jk}(d) = \left(R^{(jj)} - R^{(kk)} + 2iR^{(jk)}\right) = (R_j + iR_k)^2 \tag{6.20}$$

so that $\mathcal{B}^{12}(2) = (\mathbf{H}_{\mathbb{C}})^2 = -\mathcal{B}$. Since (A^{jk}, B^{jk}) is a conformal pair if and only if (B^{jk}, A^{jk}) is a conformal pair, the operators $\mathcal{B}^{jk}$ are essentially the same as $\mathcal{B}^{kj}$. Therefore we may assume that $j<k$.

Suppose further that d is even, say $d = 2n$, and identify $R^{2n} \cong C^n$ which decomposes orthogonally as a direct sum of coordinate 2–planes

$$R^{2n} = \bigoplus_{m=1}^n E_m, \quad E_m := \operatorname{span}\{e_{2m-1}, e_{2m}\}. \tag{6.21}$$

This gives the operators

$$\mathcal{B}_m(d) = (\mathbf{H}_{\mathbb{C}}(m))^2 = (R_{2m-1} + iR_{2m})^2, \quad m = 1, \dots n, \tag{6.22}$$

which are Beurling-Ahlfors operators on $E_m \cong \mathbb{C}$. Setting $z_m = x_{2m-1} + ix_{2m}$, $m = 1, \dots, n$ and recalling the complex derivatives

$$\partial_{z_m} = \frac{1}{2}\big(\partial_{x_{2m-1}} - i\partial_{x_{2m}}\big), \qquad \partial_{\bar{z}_m} = \frac{1}{2}\big(\partial_{x_{2m-1}} + i\partial_{x_{2m}}\big).$$

A simple computation then gives that

$$\mathcal{B}_m(d) = (R_{2m-1} + iR_{2m})^2 = 4\,\partial^2_{\bar{z}_m}\,(-\Delta)^{-1}, \tag{6.23}$$

exactly as in the complex plane $\mathbb{C}$.

These operators are the projection of the martingale transform of $M_t + iN_t \in \mathbb{C}$ obtained as above by composing the the heat extension of the complex-valued function $F = f_1 + if_2$ with space-time Brownian motion and taking the martingale transform with the matrices

$$C_m = A^{(2m-1,2m)} + iB^{(2m-1,2m)} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & A_0 + iB_0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \tag{6.24}$$

where the only nonzero 2×2 block occurs in the rows and columns $(2m-1, 2m)$ and $A_0 + iB_0$ is the 2×2 matrix in (6.16). Equivalently,

$$C_m = A^{(2m-1,2m)} + iB^{(2m-1,2m)} = \operatorname{diag}\Big(0, \dots, 0, \underbrace{A_0 + iB_0}_{(2m-1,\,2m)\text{ block}} \;0, \dots, 0\Big).$$

From the fact that

$$A_0 + iB_0 = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$$

it follows that for $z \in \mathbb{C}^n$,

$$C_m z = \big(0, \dots, 0,\; z_{2m-1} - iz_{2m},\; -iz_{2m-1} - z_{2m},\; 0, \dots, 0\big)$$

and

$$\|C_m\|_{\mathbb{C}^n \to \mathbb{C}^n} = 2.$$

In fact, due to the block structure, the following matrix which is the sum of the C_m also has norm 2.

$$C = \begin{pmatrix} A_0 + iB_0 & & & & \\ & \ddots & & & \\ & & A_0 + iB_0 & & \\ & & & \ddots & \\ & & & & A_0 + iB_0 \end{pmatrix}, \tag{6.25}$$

By (6.17) and (6.22) we have the projection operators

$$\mathcal{B}_m(d)F = (R_{2m-1} + iR_{2m})^2 F, \quad 1 \leq m \leq n. \tag{6.26}$$

Theorem 6.10. *For $m = 1, \cdots n$, set*

$$Z^m = C_m * (M + iN) \tag{6.27}$$

and

$$Z = (Z^1, Z^2, \dots Z^n) \in \mathbb{C}^n. \tag{6.28}$$

Z is a conformal martingale with the orthogonal property. Furthermore, the vector operator defined for $F : \mathbb{R}^d \to \mathbb{C}$ by

$$\mathcal{B}(d)F = (\mathcal{B}_1(d)F, \dots, \mathcal{B}_n(d)F)$$

satisfies

$$\|\mathcal{B}(d)\|_{L^p(\mathbb{R}^d;\mathbb{C})} \leq 2(p^* - 1), \quad 1 < p < \infty. \tag{6.29}$$

Remark 6.11. *This is the analogue for the Beurling-Ahlfors operators $\mathcal{B}$ on R^{2d} of inequality* (5.8) *for the Riesz transforms on $\mathbb{R}^d$. In fact, as it was already pointed out, inequality* (5.8) *also holds for complex-valued functions.*

Proof. By Theorem 6.6 each Z_m is conformal and as in (ii) of Definition 4.5,

$$Z = (Z^1, Z^2, \dots Z^n) \in \mathbb{C}^n \tag{6.30}$$

is also conformal. It remains to show that $\langle Z^m, Z^l \rangle = 0$, $l \neq m$ to satisfy property (iii) of Definition 4.5. This follows from the fact the pairs $(A^{(2m-1,2m)}, B^{(2m-1,2m)})$ act on different blocks. That is, with the matrix as above,

$$A^{(2m-1,2m)} + iB^{(2m-1,2m)} = \operatorname{diag}\Big(0, \dots, 0, \underbrace{A_0 + iB_0}_{(2m-1,\,2m)\text{ block}} \;\; 0, \dots, 0\Big),$$

the block structure guarantees that $\langle Z^m, Z^\ell \rangle_t = 0$, $\ell \neq m$. Thus Z satisfies (iii) of Definition 4.5. Note that similarly $\langle Z^m, \overline{Z^\ell} \rangle = 0$, for all $\ell \neq m$.

Since the matrix C, which is the sum of the matrices C^m, has norm 2, Burkholder's inequality (exactly as in [8, (3.4.3), p. 815]) immediately gives (6.29). □

Below we give two examples that show more explicitly how the structure of the matrices C^m give the orthogonality property $\langle Z^j, Z^\ell \rangle_t = 0$, for $\ell \neq j$.

Example 6.12 ($d = 4$)**.** *Let $W = (W^1, W^2, W^3, W^4)$ be a 4–dimensional Brownian motion and define*

$$M_t = \int_0^t H_s \cdot dW_s, \quad N_t = \int_0^t G_s \cdot dW_s,$$

where $H_s, G_s \in \mathbb{R}^4$ are predictable processes. Set

$$U_s := H_s + iG_s = (u_1, u_2, u_3, u_4) \in \mathbb{C}^4.$$

There are two pairs to consider, $(1, 2), (3, 4)$.

(1) The pair $(1,2)$**.**

$$A^{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B^{(1,2)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(A^{(1,2)} + iB^{(1,2)})U = \big(u_1 - iu_2,\, -u_2 - iu_1,\, 0,\, 0\big),$$

and

$$Z_t^1 := \int_0^t (A^{(1,2)} + iB^{(1,2)})U_s \cdot dW_s.$$

(2) The pair $(3,4)$**.**

$$A^{(3,4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad B^{(3,4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then

$$(A^{(3,4)} + iB^{(3,4)})U = \big(0,\, 0,\, u_3 - iu_4,\, -u_4 - iu_3\big),$$

and

$$Z_t^2 := \int_0^t (A^{(3,4)} + iB^{(3,4)})U_s \cdot dW_s.$$

Thus

$$d\langle Z^1, Z^2\rangle_t = \Big((A^{(1,2)} + iB^{(1,2)})U_t\Big) \cdot \Big((A^{(3,4)} + iB^{(3,4)})U_t\Big)\, dt = 0,$$

and this gives $\langle Z^1, Z^2\rangle_t = 0$*.*

Example 6.13 $(d = 6)$**.** *Let* $W = (W^1, \ldots, W^6)$ *be a* 6*–dimensional Brownian motion and let*

$$M_t = \int_0^t H_s \cdot dW_s, \quad N_t = \int_0^t G_s \cdot dW_s,$$

where $H_s, G_s \in \mathbb{R}^6$ *are predictable processes. Set*

$$U_s := H_s + iG_s = (u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{C}^6.$$

There are three pairs to consider $(1,2), (3,4), (5,6)$ *and this already shows the general proof as above. For* $m = 1, 2, 3$*, set*

$$A_m := A^{(2m-1,2m)}, \quad B_m := B^{(2m-1,2m)}, \quad Z_t^m := \int_0^t (A_m + iB_m)\, U_s \cdot dW_s.$$

The three blocks. *A direct computation (identical to the $d = 4$ case, block by block) gives*

$$\begin{aligned}(A_1 + iB_1)U &= (u_1 - iu_2,\ -u_2 - iu_1,\ 0,\ 0,\ 0,\ 0),\\ (A_2 + iB_2)U &= (0,\ 0,\ u_3 - iu_4,\ -u_4 - iu_3,\ 0,\ 0),\\ (A_3 + iB_3)U &= (0,\ 0,\ 0,\ 0,\ u_5 - iu_6,\ -u_6 - iu_5).\end{aligned}$$

Thus $(A_m + iB_m)U$ is only non-zero on the coordinate pair $\{2m - 1, 2m\}$. This gives

$$d\langle Z_m, Z_\ell\rangle_t = \Big((A_m + iB_m)U_t\Big) \cdot \Big((A_\ell + iB_\ell)U_t\Big)\, dt = 0$$

for $m \neq \ell$.

Remark 6.14. *Continuing with our assumption that $d = 2n$, we note that the operator $\mathcal{B}(d)$ is not the same as the classical Beurling-Ahlfors operator S acting on differential forms on $\mathbb{R}^d$ studied in* [41], [62], *and with martingale methods in* [23]. *Setting $S_1 = S$ acting on 1-forms Λ^1 we see from* [23, p. 234] *that S_1 is a Fourier multiplier with symbol*

$$m_{S_1}(\xi) = \left(\frac{\xi \otimes \xi - |\xi|^2 I}{|\xi|^2}\right).$$

That is,

$$S_1 = \mathcal{F}^{-1}\left(\frac{\xi \otimes \xi - |\xi|^2 I}{|\xi|^2}\right)\mathcal{F}.$$

We extend the operator $\mathcal{B}(d)$ componentwise to vector-valued functions $F = (F_1, \ldots, F_n) : \mathbb{R}^{2n} \to \mathbb{C}^n$ by

$$\mathcal{B}(d)F = \big(\mathcal{B}_1(d)F_1, \ldots, \mathcal{B}_n(d)F_n\big).$$

With this we can write

$$\mathcal{B}(d) = PS_1Q,$$

where

$$(\widehat{PS_1Qf})(\xi) = P\, m_{S_1}(\xi)\, Q\, \widehat{F}(\xi).$$

The operators P and Q are constant matrices with $|P\| = \|Q\| = 1$. Consequently, they commute with the Fourier transform. In fact, if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $z = (z_1, \ldots, z_n), \quad z_j = x_j + iy_j \ \longleftrightarrow\ (x_1, y_1, \ldots, x_n, y_n)$, $Q : \mathbb{C}^n \to \mathbb{C}^n$ is given by $Q(z_1, \ldots, z_n) = (\overline{z_1}, \ldots, \overline{z_n})$ and represented by the constant real matrix

$$Q = \operatorname{diag}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \ldots, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) \in \mathbb{R}^{2n\times 2n}$$

while

$$P = \operatorname{diag}(I_2, \ldots, I_2) \in \mathbb{R}^{2n\times 2n},$$

where each I_2 denotes the 2×2 identity matrix.

By [23, Theorem 1, p. 227]*, using the heat extension rather than Poisson extension (see also* [59]*), we have*

$$\|S_1\|_{L^p(\mathbb{R}^d;\Lambda^1)\to L^p(\mathbb{R}^d;\Lambda^1)} \leq \left(3-\frac{1}{n}\right)(p^*-1), \quad 1<p<\infty.$$

This gives

$$\|\mathcal{B}(d)\|_{L^p(\mathbb{R}^{2n});\mathbb{C}^n)} \leq \left(3-\frac{1}{n}\right)(p^*-1), \quad 1<p<\infty, \tag{6.31}$$

which is the off-diagonal version of the inequality (6.29).

We now return to more precise information on the L^p-boundedness of the individual operators $\mathcal{B}^{jk}(d)$ for $1 \leq j < k \leq d$. From (6.17) and (6.18), the contraction property of the conditional expectation on L^p and Theorems 6.5 and 6.6, it follows that

$$\|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{C_p(2)}(p^*-1), \quad 1<p<\infty, \tag{6.32}$$

when acting on real-valued functions and that

$$|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d;\mathbb{C})} \leq \frac{\sqrt{2}}{C_p(2)}(p^*-1), \quad 1<p<\infty, \tag{6.33}$$

when acting on complex-valued functions.

Furthers more, from inequalities (6.13) and (6.14) it follows that

$$\|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d)} \leq \sqrt{p(p-1)}, \quad 2\leq p<\infty, \tag{6.34}$$

when acting on real-valued functions and that

$$\|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d;\mathbb{C})} \leq \sqrt{2p(p-1)}, \quad 2\leq p<\infty, \tag{6.35}$$

when acting on complex-valued functions.

From the asymptotic behavior of $C_p(2)$ given in (i)-(iii) of Remark 5.10, we see that

$$\limsup_{p\to\infty} \frac{\|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d)}}{(p-1)} \leq 1,$$

when acting on real-valued functions and

$$\limsup_{p\to\infty} \frac{\|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d;\mathbb{C})}}{(p-1)} \leq \sqrt{2},$$

when acting on complex-valued.

For the case of the Beurling-Ahlfors operator $\mathcal{B}$ when $d=2$, the asymptotic behavior was proved in [42] prior to the bounds obtained in [12]. Their argument, restricted to the operator $\mathcal{B}$ is analytic and different from the martingale argument given above. Clearly the asymptotic behavior follows from (6.34) and (6.35). However, it is still interesting that asymptotic can be obtained just from the Burkholder (p^*-1) result.

Since,

$$\widehat{\mathcal{B}^{jk}(d)}f(\xi) = \left(\frac{i\xi_j - \xi_k}{|\xi|}\right)^2 \widehat{f}(\xi)$$

we have $\|\mathcal{B}^{jk}(d)\|_{L^2(\mathbb{R}^d)} = 1$ and the exact same interpolation argument as in [12] gives the more general inequality valid for all $d \geq 2$

$$\|\mathcal{B}^{jk}(d\|_{L^p(\mathbb{R}^d;\mathbb{C})} \leq 1.575(p^* - 1), \quad 1 < p < \infty.$$

The interpolation argument in [12] that leads to the 1.575 constant is so well optimized that it does not seem possible to improve it. For the operator acting on real-valued function the estimate (6.34) and the interpolation argument gives the bound

$$\|\mathcal{B}^{jk}(d)\|_{L^p(\mathbb{R}^d)} \leq 1.158(p^* - 1), \quad 1 < p < \infty,$$

for $\mathcal{B}^{jk}(d)$ acting on real-valued functions.

With the improvement (6.12) of Burkholder's inequality for conformal martingales given in [12], two obvious questions arise. (1) Is the inequality sharp and if not what is the sharp inequality? (2) Is there a similar improved inequality in the range $1 < p < 2$? As for (1), we would venture to say that the inequality is not sharp. Regarding (2), this and related inequalities are investigated in [31], [30] and [28]. In particular,

Theorem B ([31]). *Suppose $Y = Y_1 + iY_2$ is a conformal martingale and $X = X_1 + iX_2$ is an arbitrary martingale. Then*

$$\|Y\|_p \leq C_p\|X\|_p, \quad 1 < p \leq 2, \quad C_p = \frac{1}{\sqrt{2}}\frac{a_p}{1 - a_p}, \tag{6.36}$$

where a_p is the least positive root in the interval $(0, 1)$ of the bounded Laguerre function L_p. Furthermore, the inequality is sharp.

With this inequality the authors in [31] improve the upper bound in (6.3) for large p. More precisely, they prove that

$$\|\mathcal{B}\|_{L^p(\mathbb{C};\,\mathbb{C})} \leq 1.3992\, p, \quad \text{for} \quad p \geq 1000. \tag{6.37}$$

Exactly as above, the same result holds $\mathcal{B}^{jk}(d)$ for any $d \geq 2$. In fact one can improve the $p \geq 100$ in [31] to $p \geq 500$. By duality and estimates on a_p the authors obtain the following inequality (see [31, Theorem 10.1, p. 516])

$$\|\mathcal{B}\|_{L^p(\mathbb{C};\,\mathbb{C})} < \left(\frac{p+3}{2}\,\pi\right)^{1/(2p)} \frac{p - Q}{Q}, \quad p > 2, \tag{6.38}$$

where

$$Q = 1 - \sum_{n=2}^{\infty} \frac{(n-2)!}{(n!)^2}. \tag{6.39}$$

They estimate Q numerically to be ≈ 0.718282, [31, (9.1), p. 515]. The authors than substitute $p = 1000$ to get

$$\Big(\frac{1003}{2}\pi\Big)^{1/2000} < 1.004, \quad \text{and} \quad 1.4Q > 1.005,$$

from which (6.37) follows.

As might be expected, a bit more careful argument to estimate the quantity on the right of (6.38) will lower the cutoff point. This is in fact the case. Observe that

$$\frac{(n-2)!}{(n!)^2} = \frac{1}{n(n-1)\,n!},$$

from which it follows that $Q = 1 - (3 - e) = e - 2$. Now consider the function

$$h(p) = \frac{1}{p}\Big(\frac{p+3}{2}\,\pi\Big)^{1/(2p)}\,\frac{p-Q}{Q}, \quad p > 2.$$

Taking $\log$ and differentiating it follows that $h(p)$ is decreasing for $p \geq 4$. With a target upper bound of 1.4 testing various values of p we arrive at a safe interval where $h(p) \leq 1.4$. For example, $h(450) \approx 1.40016908$ and $h(500) \approx 1.399999$. Thus,

$$\|\mathcal{B}\|_{L^p(\mathbb{C};\mathbb{C})} \leq 1.4\,p, \quad \text{for} \quad p \geq 500. \tag{6.40}$$

In fact, numerically the values are so close that the best interval is probably in a small neighborhood of 450. Finally, $\lim_{p\to\infty} h(p) = \frac{1}{e-2} \approx 1.3922111$. These estimates brake the $\sqrt{2}$ "barrier," albeit not by much.

For a the study of topics related to those presented in this section, and specially further structures and inequalities for conformal martingales, the reader is referred to [66].

6.4. **Final Remarks.**

Remark 6.15. *While it may be possible to refine such numerical bounds further, a more compelling direction is to seek arguments that establish Iwaniec's conjecture, even for special values of* p*. This goal was a principal motivation for our study of martingale Cotlar identities beyond the classical setting. Even establishing the conjecture for* $p = 4$*, which to the best of our knowledge is still unknown, may already yield valuable insight.*

Remark 6.16. *Martingale methods, in combination with the Burkholder-Bellman techniques have produced some of the strongest bounds toward Iwaniec's conjecture and Stein's inequality. Nevertheless, a remarkable fact is that no martingale or other stochastic analysis argument is known that recovers even the simplest case* $p = 2$ *with the sharp constant* 1*; a result that follows immediately from the Fourier transform. This suggests that it is worthwhile to search for alternative approaches even for special cases of* p*.*

Remark 6.17. *A comparison between Pisier's treatment of the vector of Riesz transforms and the probabilistic representation of the Beurling-Ahlfors operator reveals the obstruction for $p = 2$. The argument in* [81] *relies on projecting onto the first Wiener chaos, but this projection fails to preserve the L^2 norm and introduces the factor $\sqrt{\pi/2}$. In the Beurling-Ahlfors setting the analogous step is the conditional expectation arising in the probabilistic representation of the operator, which is even less efficient as already at the L^2 level it incurs a loss of 2. Consequently, a direct adaptation of* [81] *to the Beurling-Ahlfors operator seems unlikely to yield sharp bounds without exploiting additional structure. Nevertheless, such an approach may still lead to improvements over the currently known estimates and provide further insight into the problem.*

Remark 6.18. *Motivated by Pisier's treatment of higher order Riesz transforms leads us to further speculate about a possible adaptation of this method to the Beurling-Ahlfors operator. Unlike the Riesz transforms whose symbols are linear in ξ, the multiplier for $\mathcal{B}$ is quadratic in ξ. In the Gaussian framework underlying the argument in* [81]*, linear dependence on ξ corresponds naturally to the first Wiener chaos. The quadratic structure suggests that the appropriate analogue of the projection onto the first Wiener chaos would be a projection onto a suitable subspace of the second Wiener chaos. This approach would be similar to Remark* [81, Remark, p.496] *concerning higher order Riesz transforms.*

From (5.20) *we have that the second Wiener chaos is*

$$\mathcal{H}_2 = \operatorname{span}\{H_\alpha : |\alpha| = 2\}$$

with bases

$$\{H_2(x_1) = x_1^2 - 1,\ H_2(x_2) = x_2^2 - 1,\ H_1(x_1)\, H_1(x_2) = x_1 x_2\}.$$

Thus

$$\bar{\xi}^2 = (\xi_1 - i\xi_2)^2 = (\xi_1^2 - \xi_2^2) - 2i\xi_1\xi_2 \in \mathcal{H}_2.$$

Similarly,

$$(\xi_1, \xi_2) \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (\xi_1 + i\xi_2)^2 \in \mathcal{H}_2,$$

and this shows that the same quadratic combination arises from the matrix $A_0 + iB_0$ appearing in the martingale representation. Thus both the Gaussian and martingale viewpoints point to the second-chaos structure.

It should be noted that this approach, if applicable, will produce an additional factor of $\frac{1}{(p-1)}$ in the range $1 < p < 2$ in place of $\frac{1}{\sqrt{p-1}}$ arising from hypercontractivity. In addition, as pointed out in [81, Remark, p. 497]*, projecting onto the k-Wiener chaos to study norms of higher order Riesz transforms requires k to be odd, similarly to the method of rotations for powers of $\mathbf{H}_{\mathbb{C}}$.*

Acknowledgments. Part of the material presented here was prepared during the Fall 2024 semester, while the author was teaching an advanced graduate course in harmonic analysis. Although none could be included in the course due to time constraints, the author nevertheless wishes to acknowledge the students' stimulating discussions and presentations on related topics. The author is grateful to Prabhu Janakiraman for the insightful discussions on Iwaniec's conjecture over the years, particularly during the past two years, and the many comments and corrections on earlier versions of these notes.

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DEPARTMENT OF MATHEMATICS. PURDUE UNIVERSITY
Email address: banuelos@purdue.edu