SHARP ℓ^p INEQUALITIES FOR DISCRETE SINGULAR INTEGRALS ON THE LATTICE $\mathbb{Z}^d$

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In memory of Richard F. Gundy (1933-2023)

ABSTRACT. This paper investigates higher dimensional versions of the longstanding conjecture verified in [10] that the ℓ^p-norm of the discrete Hilbert transform on the integers is the same as the L^p-norm of the Hilbert transform on the real line. It computes the ℓ^p-norms of a family of discrete operators on the lattice $\mathbb{Z}^d$, $d \geq 1$. They are discretizations of a new class of singular integrals on $\mathbb{R}^d$ that have the same kernels as the classical Riesz transforms near zero and similar behavior at infinity. The discrete operators have the same p-norms as the classical Riesz transforms on $\mathbb{R}^d$. They are constructed as conditional expectations of martingale transforms of Doob h-processes conditioned to exit the upper–half space $\mathbb{R}^d \times \mathbb{R}_+$ only on the lattice $\mathbb{Z}^d$. The paper also presents a discrete analogue of the classical method of rotations which gives the norm of a different variant of discrete Riesz transforms on $\mathbb{Z}^d$. Along the way a new proof is given based on Fourier transform techniques of the key identity used to identify the norm of the discrete Hilbert transform in [10]. Open problems are stated.

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1. Introduction and Statement of Results

The probabilistic representation à la Gundy–Varopoulos [38] of the classical Riesz transforms and other singular integrals and Fourier multipliers as conditional expectations (projections) of stochastic integrals, in combination with the sharp martingale inequalities of Burkholder [21] and their versions for orthogonal martingales [16] and non-symmetric transforms [13, 26], has proven to be a powerful tool in obtaining sharp, or near sharp, L^p-bounds for these operators in a variety of geometric settings. A particular feature of these techniques is that they give L^p-bounds independent of the geometry of the ambient space. Most notably, the bounds are independent of the dimension. For example, in [16] such techniques were used to show that the L^p-norm, $1 < p < \infty$, of the Riesz transforms on $\mathbb{R}^d$, $d > 1$, is the same as that of the Hilbert transform on $\mathbb{R}$ found by S. Pichorides [57] and B. Cole (see [35]), and to obtain the first explicit bounds for the Beurling–Ahlfors operator. The former was first proved using the method of rotations in [44]. For some history on norm estimates for the Beurling–Ahlfors transform motivated by the celebrated 1982 conjecture of T. Iwaniec [43] and the current best known bound, see [9] and the survey article [5].

One advantage of the martingale approach is that it immediately extends to geometric and analytic settings well beyond $\mathbb{R}^d$, including the infinite dimensional case of Wiener space and other semigroups such as those of Lévy processes and discrete Laplacian on groups. The interest on dimension free estimates for Riesz transforms and other operators in harmonic analysis was initiated by the results and questions raised in Stein [66] and Meyer [53]. For some of the now vast literature on dimension free and sharp bounds for Riesz transforms and Fourier multipliers in a variety of geometric and analytic settings and many uses of probabilistic tools, we refer the reader to [2, 5–8, 12–15, 17, 23, 24, 27–34, 36, 39, 49–51, 54–56, 63, 71] and references contained therein.

1.1. **The probabilistic discrete Hilbert transform on $\mathbb{Z}$.** In his celebrated 1928 paper [64], M. Riesz solved a problem of considerable interest to the analysis community at the time by showing that the Hilbert transform

$$(1.1) \qquad Hf(x) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-y)}{y} dy, \quad f \in L^p(\mathbb{R}),$$

is a bounded operator on $L^p(\mathbb{R})$, $1 < p < \infty$. For some history on this problem and Riesz's solution in 1925 before its publication in 1928, we refer the reader to M. Cartwright's article "Manuscripts of Hardy, Littlewood, Marcel Riesz and Titchmarsh," [25]. In his paper Riesz also showed that the boundedness of H on $L^p(\mathbb{R})$ implies the boundedness of the discrete version H_{dis} on $\ell^p(\mathbb{Z})$, where the latter is defined by the convolution with $\{\frac{1}{\pi m}, m \in \mathbb{Z} \setminus \{0\}\}$. That is,

$$(1.2) \qquad H_{\mathrm{dis}}f(n) = \frac{1}{\pi} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{f(n-m)}{m}, \quad f \in \ell^p(\mathbb{Z}).$$

In fact, Riesz showed that the operator norms satisfy

$$(1.3) \qquad \|H\|_{L^p \to L^p} \le \|H_{\mathrm{dis}}\|_{\ell^p \to \ell^p}, \quad \|H_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \le C\|H\|_{L^p \to L^p},$$

for some constant C.

The discrete Hilbert transform was introduced in 1909 by D. Hilbert who verified its boundedness on $\ell^2(\mathbb{Z})$. Proving that the operator norms of H and H_{dis}, $1 < p < \infty$, are the same had been a long-standing open problem motivated in part by an erroneous proof of E. C. Titchmarsh in 1926, [69, 70].

Throughout this paper we will use the notation

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}, \quad 1 < p < \infty.$$

With this notation, the Pichorides [57] and Cole [35] results say that

$$(1.4) \qquad \|H\|_{L^p \to L^p} = \cot\left(\frac{\pi}{2p^*}\right) = \begin{cases} \tan\left(\frac{\pi}{2p}\right), & 1 < p \le 2 \\ \cot\left(\frac{\pi}{2p}\right), & 2 \le p < \infty. \end{cases}$$

When p is of the form 2^j or $2^j/(2^j-1)$, $j \in \mathbb{N}$, the result had been known for several years. For such p, it is shown in [47] that (1.4) holds for H_{dis}. For further history and references related to this problems, see [2, 11, 27–30, 40, 41, 47]. In [10], it is proved that for all $1 < p < \infty$,

$$(1.5) \qquad \|H_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \le \cot(\pi/(2p^*))$$

and hence, together with the first inequality in (1.3),

$$(1.6) \qquad \|H_{\mathrm{dis}}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}.$$

The proof of (1.5) rests on a modification of the Gundy–Varopoulos techniques using occupation time formulas as in [3] for the Doob h-process where the harmonic function

h corresponds to the periodic Poisson kernel. This construction leads to yet a third operator $T_{\mathbb{H}}$, which we call the *probabilistic discrete Hilbert transform*.

$$(1.7) \qquad T_{\mathbb{H}}f(n) = \sum_{m \in \mathbb{Z}\setminus\{0\}} \mathbb{K}_{\mathbb{H}}(m)f(n-m),$$

where

$$(1.8) \qquad \mathbb{K}_{\mathbb{H}}(m) = \frac{1}{\pi m}\left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 m^2)\sinh^2(y)}\,dy\right).$$

Although, as (1.8) clearly shows, this construction does not lead to an exact representation of the discrete Hilbert transform, unlike the classical case for the continuous Hilbert transform on $\mathbb{R}$ and the Gundy–Varopoulos extension for Riesz transforms on $\mathbb{R}^d$, the martingale inequalities from [16] show that for $1 < p < \infty$,

$$(1.9) \qquad \|T_{\mathbb{H}}\|_{\ell^p \to \ell^p} \le \cot(\pi/(2p^*)).$$

Furthermore, it is shown in [10] that there exists a probability kernel $\mathcal{P}$ on $\mathbb{Z}$ such that

$$(1.10) \qquad H_{\text{dis}}f(n) = (T_{\mathbb{H}} * \mathcal{P})f(n).$$

Here $*$ denotes the convolution. From this,

$$\|H_{\text{dis}}\|_{\ell^p \to \ell^p} \le \|T_{\mathbb{H}}\|_{\ell^p \to \ell^p} \le \cot(\pi/(2p^*))$$

and (1.6) follows.

In this paper we give a new proof of (1.10); see Section 8, Theorem 8.1. The proof is based on Fourier transform techniques.

Before discussing the higher dimensional case, we point out that there are other variants of the discrete Hilbert transform on $\mathbb{Z}$. These include the one introduced by Titchmarsh in [69], called the *Riesz–Titchmarsh operator*, for which the conjecture that their ℓ^p norms coincide with that of H remains open. For a detailed discussion of the different variants of discrete Hilbert transforms and a proof of the conjecture for $p = 2n$ or its conjugate $p = \frac{2n}{2n-1}$, $n \in \mathbb{N}$, for the Riesz–Titchmarsh operator, we refer the reader to [11]. While the argument in [11] does not use probability, it does use the fact that (1.6) holds for all $1 < p < \infty$.

1.2. **The probabilistic discrete Riesz transforms on $\mathbb{Z}^d$.** In their celebrated paper [22] Calderón and Zygmund remark that Riesz's argument for the second inequality in (1.3) applies to show that the boundedness of singular integrals on $L^p(\mathbb{R}^d)$ implies the boundedness of their discrete versions defined on $\mathbb{Z}^d$ in the same manner as the discrete Hilbert transform. More precisely, let

$$Tf(z) = p.v. \int_{\mathbb{R}^d} K(z)f(x-z)dz, \quad f : \mathbb{R}^d \to \mathbb{R}$$

be a Calderón-Zygmund singular integral (K as in (6.3)). Then the discrete operator T_{dis} defined by

$$(1.11) \qquad T_{\text{dis}}(f)(n) = \sum_{m \in \mathbb{Z}^d\setminus\{0\}} K(m)f(n-m), \quad f : \mathbb{Z}^d \to \mathbb{R}$$

is bounded on $\ell^p(\mathbb{Z}^d)$ and

$$\|T_{\text{dis}}\|_{\ell^p \to \ell^p} \le C\|T\|_{L^p \to L^p},$$

for some constant C; see [22, pg. 138]. In particular, if we consider the Riesz transforms on $\mathbb{R}^d$

$$(1.12) \qquad R^{(k)}f(x) = p.v. \int_{\mathbb{R}^d} K^{(k)}(z)f(x-z)\,dz, \quad k = 1, 2, \ldots d,$$

with

$$(1.13) \qquad K^{(k)}(z) = c_d \frac{z_k}{|z|^{d+1}}, \quad c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}$$

and their discrete versions

$$(1.14) \qquad R_{\mathrm{dis}}^{(k)}f(n) = c_d \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \frac{m_k}{|m|^{d+1}} f(n-m),$$

it holds that for all $1 < p < \infty$,

$$\|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p} \le C\|R^{(k)}\|_{L^p \to L^p}, \quad 1 < p < \infty.$$

for some constant C.

These operators are the classical extensions of the Hilbert transform to several dimensions. It is shown in [44] using the method or rotations, and in [16] using martingale inequalities, that

$$(1.15) \qquad \|R^{(k)}\|_{L^p \to L^p} = \|H\|_{L^p \to L^p} = \cot(\pi/(2p^*))$$

From Proposition 6.1 below it follows that

$$(1.16) \qquad \|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p} \le \|R^{(k)}\|_{L^p \to L^p} + C_d = \cot(\pi/(2p^*)) + C_d,$$

where C_d is a constant depending only on d. Furthermore, in Theorem 6.2 a sharp lower bound for singular integrals with homogeneous kernels is proved from which it follows that

$$(1.17) \qquad \cot(\pi/(2p^*)) = \|R^{(k)}\|_{L^p \to L^p} \le \|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p}.$$

This gives

$$(1.18) \qquad \cot(\pi/(2p^*)) \le \|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p} \le \cot(\pi/(2p^*)) + C_d$$

Given the success of probabilistic techniques in deriving sharp estimates for Riesz transforms in general geometric settings, as discussed above, together with the interest on discrete analogues of many classical operators in harmonic analysis as studied in [20, 52, 58–62, 67, 68] (and many other references contained therein), together with the results in [10], the following questions naturally arise:

(1) Can the construction of the probabilistic operators in [10] be carried out in higher dimension to obtain a collection of operators on $\mathbb{Z}^d$, $d > 1$, which are closely related to the classical Riesz transforms in (1.14), that have $\ell^p(\mathbb{Z}^d)$-norms independent of d and that could be used to show that $\|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p}$ is independent of d or even that it equals $\cot(\pi/(2p^*))$, as suggested by (1.18)?
(2) Do the probabilistic discrete operators arise as the discretization of Calderón-Zygmund singular integrals on $\mathbb{R}^d$–as is the case of (1.14)?

(3) Given the important role of the method of rotations in obtaining sharp or near sharp estimates for Riesz transforms and related singular integrals on $\mathbb{R}^d$ (see for example [44]), it is natural to ask: is there a discrete "method of rotations" that would reduce the $\ell^p(\mathbb{Z}^d)$ boundedness of $R_{\mathrm{dis}}^{(k)}$ to that of $H_{\mathrm{dis}}(\mathbb{Z})$?

The goal of this paper is to address these questions as we now explain. The periodic Poisson kernels $h(x, y)$, $x \in \mathbb{R}^d, y > 0$, is obtained by summing the translations of the classical Poisson kernel $p_n(x, y) = p(x - n, y)$ over $n \in \mathbb{Z}^d$ as in (4.2). Using this positive harmonic function we construct a Doob h-process in the upper half-space which exits only on the lattice $\mathbb{Z}^d$. Taking conditional expectations of martingale transforms of stochastic integrals built with this process, we construct a large collection of operators on $\mathbb{Z}^d$, denoted by T_A, where A is any $(d + 1) \times (d + 1)$ matrix with variable entries and finite quadratic norm $\|A\|$ as defined in (4.12). Applying the martingale inequalities in [21] and [16], leads to the operator bounds $\|T_A\|_{\ell^p \to \ell^p} \le \|A\|(p^* - 1)$ for all A and $\|T_A\|_{\ell^p \to \ell^p} \le \|A\| \cot(\pi/(2p^*))$, if in addition A has the property that Av is orthogonal to v, for all $v \in \mathbb{R}^{d+1}$. The kernels for the T_A's, (not necessarily of convolution type) are computed in Theorem 5.1. Of particular interest in this construction is the case of the sequence of matrices $\mathbb{H}^{(k)} = (a_{ij}^{(k)})$, $k = 1, 2, \ldots d$, given by

$$(1.19) \qquad a_{ij}^{(k)} = \begin{cases} -1, & i = k, j = d + 1 \\ 1, & i = d + 1, j = k \\ 0, & \text{otherwise.} \end{cases}$$

These matrices have the orthogonality property, $\mathbb{H}^{(k)}v \cdot v = 0$ for all $v \in \mathbb{R}^{d+1}$, and $\|\mathbb{H}^{(k)}\| \le 1$. Hence the corresponding operators $T_{\mathbb{H}^{(k)}}$ satisfy

$$(1.20) \qquad \|T_{\mathbb{H}^{(k)}}\|_{\ell^p \to \ell^p} \le \cot(\pi/(2p^*)), \quad k = 1, 2, \ldots, d.$$

In this case the kernels (see (7.8) in the proof of Theorem 7.2) are given by

$$(1.21) \qquad \mathbb{K}_{\mathbb{H}^{(k)}}(m) = -4 \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h(x, y)} \frac{\partial p_0}{\partial x_k}(x, y) \frac{\partial}{\partial y}(y p_m)(x, y) \, dy dx,$$

and

$$(1.22) \qquad T_{\mathbb{H}^{(k)}}(f)(n) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \mathbb{K}_{\mathbb{H}^{(k)}}(n - m) f(m).$$

The choice of the $\mathbb{H}^{(k)}$'s is motivated by the fact that using the standard Brownian motion in $\mathbb{R}_+^{d+1}$ killed upon exiting on $\mathbb{R}^d$, these are the matrices used for the probabilistic representation of the classical Riesz transforms. This was originally done by Gundy and Varopoulos [38] (see Section 6.2 below) and subsequently by many other authors in different geometric settings. Indeed, the Gundy-Varopoulos construction gives operators whose kernels are as in (1.21) with h replaced by 1 and $n \in \mathbb{Z}^d$ by $z \in \mathbb{R}^d$. In fact,

$$(1.23) \qquad -4 \int_{\mathbb{R}^d} \int_0^\infty \frac{\partial p_0(x, y)}{\partial x_k} \frac{\partial}{\partial y}(y p_z(x, y)) \, dy dx = \frac{c_d z_k}{|z|^{d+1}}, \quad z \in \mathbb{R}^d \setminus \{0\},$$

where as before $p_z(x, y) = p(x - z, y)$ is the Poisson kernel. For this computation, see (9.10) below.

We call the operators $T_{\mathbb{H}^{(k)}}$, $k = 1, 2, \ldots, d$, the *probabilistic discrete Riesz transforms*. When $d = 1$ there is only one matrix $\mathbb{H}^{(1)}$ that we simply denote by $\mathbb{H}$ and the corresponding operator $T_{\mathbb{H}}$ is the *probabilistic discrete Hilbert transform* (1.7) studied in [10]. In this case the kernel can be further simplified to (1.8). This simplification is due to the fact that when $d = 1$ we have the simple closed formula for periodic Poisson kernel h given by (4.7). This gives a very nice and quite simple formula for $1/h$ (see (8.2)) from which many explicit computations can be done. Such a closed formula is not available for $d > 1$. Instead, various properties of the function $h(x, y)$ (as in Lemma 4.1) are used for many of the computations in this paper.

While at present there is no analogue to (1.10) for $d > 1$ (see Question 8.7 below), after a scaling and limiting arguments (Lemmas 7.6 and 7.7), the probabilistic Riesz transforms are closely related to the classical Riesz transforms. This relationship can be exploited to prove the lower bound

$$(1.24) \qquad \cot(\pi/(2p^*)) \le \|T_{\mathbb{H}^{(k)}}\|_{\ell^p \to \ell^p}.$$

This together with (1.20) shows that the ℓ^p-norm of $T_{\mathbb{H}^{(k)}}$ are not only independent of the dimension but equal the L^p-norm of $R^{(k)}$, see Theorem 7.5.

Given our discussion above the following kernels on $\mathbb{R}$ and $\mathbb{R}^d$, $d > 1$, respectively, naturally arise:

$$(1.25)$$
$$\mathbb{K}_{\mathbb{H}}(z) = \frac{1}{\pi z} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 z^2) \sinh^2(y)}\, dy\right) \mathbb{1}_{\{|z| \ge 1\}}(z) + \frac{1}{\pi z} \mathbb{1}_{\{|z| < 1\}}(z), \quad z \in \mathbb{R} \setminus \{0\},$$

$$(1.26) \qquad \mathbb{K}_{\mathbb{H}^{(k)}}(z) = \left(-4 \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h(x, y)} \frac{\partial p_0}{\partial x_k} \frac{\partial}{\partial y}(y p_z)\, dy dx\right) \mathbb{1}_{\{|z| \ge 1\}}(z)$$
$$+ c_d \frac{z_k}{|z|^{d+1}} \mathbb{1}_{\{|z| < 1\}}(z), \quad z \in \mathbb{R}^d \setminus \{0\}, \ d > 1, \ k = 1, 2, \ldots, d.$$

It is proved in Section 9, Theorem 9.8, that these kernels satisfy the Hörmader condition. The principal value convolution operators $Tf(z) = \mathbb{K}_{\mathbb{H}} * f(z)$ and $T^{(k)}f(z) = \mathbb{K}_{\mathbb{H}^{(k)}} * f(z)$ are bounded on $L^p(\mathbb{R})$ and $L^p(\mathbb{R}^d)$, $d > 1$, $1 < p < \infty$. Furthermore, it is proved in Section 9 (see Theorem 9.5) that

$$\cot(\pi/(2p^*)) \le \|T\|_{L^p \to L^p} \le \cot(\tfrac{\pi}{2p^*}) + 0.09956,$$

and

$$\cot(\pi/(2p^*)) \le \|T^{(k)}\|_{L^p \to L^p} \le \cot(\tfrac{\pi}{2p^*}) + C_d$$

where C_d depends only on d.

The probabilistic discrete Hilbert and Riesz transforms on the lattice $\mathbb{Z}^d$ are the discretization of these Calderón-Zygmund operators on $\mathbb{R}^d$. We call the operators T and $T^{(k)}$ the *probabilistic continuous Hilbert and Riesz transforms*, respectively. We conjecture (Conjecture 9.6) that their L^p-norms are not only independent of the dimension but that in fact they are equal to $\cot(\pi/(2p^*))$.

Remark 1.1. *It is important to note that these discrete singular integrals á la Hilbert, Riesz and Calderón and Zygmund, do not arise as "genuine" Riesz transforms of semigroups associated with discrete or semi-discrete Laplacians for which many results exist*

and to which the "usual" Gundy–Varopoulos construction as outlined in Section 6.2 applies. For some of these applications we refer the reader to [1, 2, 29].

Concerning a "method of rotations" in the discrete setting, Section 10 presents an analogue motivated by differently defined discrete Riesz transforms $\mathcal{R}^{(k)}$ on $\mathbb{Z}^d$. It gives a representation of these discrete Riesz transforms in terms of the discrete Hilbert transform H_{dis} from which the sharp estimates for their norms is obtained, Theorem 10.5.

We summarize the main results of this paper in the following Theorem.

Theorem 1.2. *Let $R^{(k)}$ be the classical Riesz transforms in* (1.12), *$R^{(k)}_{\mathrm{dis}}$ the discrete Riesz transforms in* (1.14), *$T_{\mathbb{H}^{(k)}}$ the probabilistic discrete Riesz transforms in* (1.21) *and $\mathcal{R}^{(k)}$ the Riesz transforms obtained by the method of rotations in* (10.3). *Fix $1 < p < \infty$.*

(i) For $d \geq 2$ and $k = 1, \ldots, d$,

$$(1.27) \qquad \|T_{\mathbb{H}^{(k)}}\|_{\ell^p \to \ell^p} = \|\mathcal{R}^{(k)}\|_{\ell^p \to \ell^p} = \|R^{(k)}\|_{L^p \to L^p} = \cot(\tfrac{\pi}{2p^*}),$$

and

$$(1.28) \qquad \cot(\tfrac{\pi}{2p^*}) \leq \|R^{(k)}_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \leq \cot(\tfrac{\pi}{2p^*}) + C_d.$$

The constant C_d depends only on d.

(ii) The singular integral operators T and $T^{(k)}$ on $\mathbb{R}$ and $\mathbb{R}^d$, $d > 1$, with kernels $\mathbb{K}_{\mathbb{H}}$ and $\mathbb{K}_{\mathbb{H}^{(k)}}$, satisfy the Hörmader condition. Further,

$$(1.29) \qquad \cot(\pi/(2p^*)) \leq \|T\|_{L^p \to L^p} \leq \cot(\pi/(2p^*)) + 0.09956$$

and

$$(1.30) \qquad \cot(\pi/(2p^*)) \& \leq \|T^{(k)}\|_{L^p \to L^p} \leq \cot(\tfrac{\pi}{2p^*}) + C_d$$

The constant C_d depends only on d.

(iii) When $d = 1$,

$$(1.31) \qquad R^{(1)} = H, \quad R^{(1)}_{\mathrm{dis}} = \mathcal{R}^{(1)} = H_{\mathrm{dis}}, \quad T_{\mathbb{H}^{(1)}} = T_{\mathbb{H}}.$$

The p-norm of these operators all equal $\cot(\tfrac{\pi}{2p^*})$.

Remark 1.3. *We note that (iii) is the content of [10]. We state it here to more clearly give a global picture for all $d \geq 1$. The contribution of this paper to (iii) is the new proof of the key identity* (1.10) *based on Fourier transform techniques which has the potential for extending to higher dimensions. (See also Remark 7.8 for a derivation of* (1.8).*) This motivates Question 8.7. If this question is answered in the affirmative it would resolve Conjecture 6.4.*

It is perhaps interesting to end this introduction with the following insightful comment from Stein [66] who ends his paper with the remark:

> *"The above results raise the following general question. Can one find an appropriate infinite-dimensional formulation of (that part of) harmonic analysis in $\mathbb{R}^n$ which displays in a natural way the above uniformity in n? A related question is to study the "limit as $n \to \infty$" of the above results, insofar as such limits may have a meaning. One might guess that a further understanding of these questions would involve, among*

> *other things, notions from probability theory: i.e. Brownian motion and*
> *possibly some variant of the central limit theorem."*

The now vast literature on probabilistic techniques for problems related to this topic, some of it cited in this paper, proves that his "guess" was indeed correct.

2. Organization of the paper

- Section 4.1 introduces the *periodic Poisson kernel h* on $\mathbb{R}^d$, $d \geq 1$, from which we will define the Doob h-process used throughout the paper and derive some of its basic properties.

- Section 4.2 defines the h-harmonic extension of the function $f : \mathbb{Z}^d \to \mathbb{R}$ and discusses a connection to an interesting problem of Magyar, Stein, and Wainger in [52], see Remark 4.4.

- Section 4.3 defines the Doob h-process associated with the function h, the martingale transforms and recalls the relevant martingale inequalities.

- Section 5.1 defines the projection operators, denoted by T_A, as the conditional expectations of martingale transforms. Theorems 5.1 and 5.5 compute their kernels and prove their boundedness on $\ell^p(\mathbb{Z}^d)$ with the same bounds as those in the martingale inequalities.

- Section 6.1 considers the discrete Calderón–Zygmund operators given by (1.11) and proves their boundedness properties. This is done in Proposition 6.1 and Theorem 6.2. From these, (ii) of Theorem 1.2 follows. Conjecture 6.4, and the weaker Problem 6.5, on the norm of the discrete Riesz transforms are formulated in this section.

- Section 7.1 shows that the $\ell^p(\mathbb{Z}^d)$-norms of the probabilistic discrete Riesz transforms on $\mathbb{Z}^d$ are the same as the $L^p(\mathbb{R}^d)$-norms of the classical Riesz transforms. This is done in Theorem 7.5. From this it follows that the p-norms of the classical Riesz transforms on $L^p(\mathbb{R}^d)$, the classical Hilbert transform on $L^p(\mathbb{R})$, the discrete Hilbert transform on $\ell^p(\mathbb{Z})$, and probabilistic discrete Riesz transforms on $\ell^p(\mathbb{Z}^d)$ are all equal to $\cot(\pi/(2p^*))$, verifying (i) in Theorem 1.2 for all the the operators with the exception of $\mathcal{R}^{(k)}$.

- Section 8 computes the Fourier transform of the probabilistic discrete Hilbert transform. This allows for a new proof of the "Key Lemma 1.3" in [10] which shows that the discrete Hilbert transform H_{dis} is the convolution of the probabilistic discrete Hilbert transform with a probability kernel, see Theorem 8.1. The proof here is based on the Fourier transform techniques. The natural question for $\mathbb{Z}^d$, $d \geq 2$ is stated at the end of this section, see Question 8.7.

- Section 9 shows that replacing the discrete variable $n \in \mathbb{Z}^d$ by the continuous variable $z \in \mathbb{R}^d$ in the kernel for the probabilistic discrete Riesz transforms, and after a modification near zero that does not affect the discrete operator, give singular integral operators satisfying the Hörmader condition and are bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$ with their norms satisfying (1.29) and (1.30). See Theorems 9.1, 9.5, 9.8, and Corollary 9.3.

- Section 10 discusses a "method of rotation" by constructing certain discrete Riesz transforms on $\mathbb{Z}^d$ motivated by the classical ones and verifying that these Riesz transforms have the same norms as the discrete Hilbert transform H_{dis} and the probabilistic discrete Riesz transforms $T_{\mathbb{H}(k)}$, see Theorem 10.5.

- Section 11 presents some numerical calculations comparing the relative sizes of the kernels for the discrete Riesz transforms, the probabilistic discrete Riesz transforms and the discrete Riesz transforms constructed in the method of rotations.

3. NOTATION

The Fourier transform of a function f on $\mathbb{R}^d$ is denoted by $\widehat{f}$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \quad \text{for } \xi \in \mathbb{R}^d.$$

For a function $f : \mathbb{Z}^d \to \mathbb{R}$, the Fourier transform is denoted by $\mathcal{F}(f)$, where

$$\mathcal{F}(f)(\xi) = \sum_{n \in \mathbb{Z}^d} f(n) e^{-2\pi i n \cdot \xi} \quad \text{for } \xi \in Q := [-\tfrac{1}{2}, \tfrac{1}{2})^d.$$

Here, Q is often called the fundamental cube.

The standard notations $\|f\|_{L^p}$ and $\|f\|_{\ell^p}$ are used for the p-norm of functions in $L^p(\mathbb{R}^d)$ and $\ell^p(\mathbb{Z}^d)$, respectively. $\|T\|_{L^p \to L^p}$ will denote the operator norm of $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, and similarly $\|T\|_{\ell^p \to \ell^p}$ for the operator norm of $T : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$.

The gradient and Laplacian of functions $u(x, y)$ on the upper half-space $\mathbb{R}^d \times \mathbb{R}_+ = \{(x, y) : x \in \mathbb{R}^d, y > 0\}$ are denoted by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_d}, \frac{\partial u}{\partial y} \right) = \left(\nabla_x u, \ \frac{\partial u}{\partial y} \right)$$

and

$$\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i{}^2} + \frac{\partial^2 u}{\partial y^2} = \Delta_x u + \frac{\partial u^2}{\partial y^2},$$

respectively. By abuse of notation, for $f : \mathbb{R}^d \to \mathbb{R}$ we will still use $\Delta f = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i{}^2}$ to denote its Laplacian on $\mathbb{R}^d$.

Throughout the paper, $C_1, C_2, \ldots$ or C_d will denote constants that depend only on d and whose value may change from line to line.

4. PRELIMINARIES

4.1. **The periodic Poisson kernel.** Let $d \geq 1$. The Poisson kernel for the upper half-space $\mathbb{R}^d \times \mathbb{R}_+$ is given by

$$(4.1) \qquad p(x, y) = \frac{c_d y}{(|x|^2 + y^2)^{\frac{d+1}{2}}}, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}_+, \quad c_d = \Gamma\left(\frac{d+1}{2} \right) \pi^{-\frac{d+1}{2}}.$$

For $x, z \in \mathbb{R}^d$ and $y \in \mathbb{R}_+$, we set $p_z(x, y) = p(x - z, y)$. Since $\Delta p_z(x, y) = 0$ and $p_z(x, y) > 0$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+$ and $z \in \mathbb{R}^d$, we see that the function h defined by

$$(4.2) \qquad\qquad\qquad h(x, y) = \sum_{n \in \mathbb{Z}^d} p_n(x, y)$$

is also positive and harmonic. In addition, it is periodic in x in the sense that $h(x + m, y) = h(x, y)$ for all $m \in \mathbb{Z}^d$. We call the function $h(x, y)$ the periodic Poisson kernel. The following properties of $h(x, y)$ will be used frequently in the sequel.

Lemma 4.1. *We have* $\lim_{y \to \infty} h(x, y) = 1$ *uniformly in* $x \in \mathbb{R}^d$. *In particular, for each* $y_0 > 0$, *there exist constants* $C_1, C_2 > 0$ *such that* $C_1 \le h(x, y) \le C_2$, *for all* $x \in \mathbb{R}^d$ *and* $y \ge y_0$.

Proof. Recall that $Q = [-\frac{1}{2}, \frac{1}{2})^d$. For $x \in \mathbb{R}^d$ and $y > 0$, we have

$$1 = \int_{\mathbb{R}^d} p(x - z, y) \, dz = \sum_{n \in \mathbb{Z}^d} \int_{n+Q} p(x - z, y) \, dz = \sum_{n \in \mathbb{Z}^d} \int_Q p_n(x - z, y) \, dz.$$

We will estimate the quantity

$$p_n(x, y) - \int_Q p_n(x - z, y) \, dz = \int_Q (p_n(x, y) - p_n(x - z, y)) \, dz.$$

All constants $C_1, C_2, \dots$ below are positive and depend only on d. Observe that we have

$$|\nabla_x p(x, y)| \le C_1 \frac{y}{(|x|^2 + y^2)^{\frac{d}{2}+1}}$$

for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+$. Furthermore, if $x \in \mathbb{R}^d$, $y \ge 1$, $z \in Q$ and $n \in \mathbb{Z}^d$, then

$$|x - n - z|^2 + y^2 \ge C_2(|x - n|^2 + y^2).$$

It follows that

$$\begin{aligned}
|\nabla_x p_n(x - z, y)| &\le C_1 \frac{y}{(|x - n - z|^2 + y^2)^{\frac{d}{2}+1}} \\
&\le C_3 \frac{y}{(|x - n|^2 + y^2)^{\frac{d}{2}+1}} \\
&\le C_4 \frac{p_n(x, y)}{(|x - n|^2 + y^2)^{\frac{1}{2}}} \le C_4 \frac{p_n(x, y)}{y}.
\end{aligned}$$

Thus,

$$|p_n(x, y) - p_n(x - z, y)| \le C_4 \frac{p_n(x, y)}{y} \times |z| \le C_5 \frac{p_n(x, y)}{y}.$$

We conclude that

$$\begin{aligned}
|h(x, y) - 1| &= \left| \sum_{n \in \mathbb{Z}^d} \int_Q (p_n(x, y) dz - p_n(x - z, y)) dz \right| \\
&\le \sum_{n \in \mathbb{Z}^d} \int_Q |p_n(x, y) dz - p_n(x - z, y)| dz \\
&\le \sum_{n \in \mathbb{Z}^d} \int_Q C_5 \frac{p_n(x, y)}{y} \, dz = \frac{C_5 h(x, y)}{y}.
\end{aligned}$$

This proves the first statement of the lemma. The second assertion (ii) follows from (i) and the fact that $h(x, y)$ is positive, continuous, and periodic in x. $\square$

Clearly, $h(x,y) \geq p(x,y)$, and thus for some constant C_1 depending only on d we have $h(x,y) \geq C_1 y$ whenever $x \in Q$ and $y \in (0,1)$. Since $h(x,y)$ is periodic in x with period 1, the same estimate holds for all $x \in \mathbb{R}^d$, and by combining this inequality with Lemma 4.1, we find that

$$(4.3) \qquad\qquad h(x,y) \geq C_2 \min\{1,y\}$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}_+$. Similarly, we have $h(x,y) \geq p(x,y) \geq C_3 \frac{y}{\sqrt{x^2+y^2}}$ when $x^2 + y^2 < 1$ and $y \in (0,1)$. For all other (x,y) the last estimate is weaker than (4.3) (up to a constant factor), so we conclude that

$$(4.4) \qquad\qquad h(x,y) \geq C_4 \frac{y}{\sqrt{x^2+y^2}}$$

holds for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}_+$.

We recall the Poisson summation formula.

Proposition 4.2 ([37, Theorem 3.1.17]). *Suppose that $f, \widehat{f} \in L^1(\mathbb{R}^d)$ and*

$$|f(x)| + |\widehat{f}(x)| \leq C(1 + |x|)^{-n-\delta}$$

for some $C, \delta > 0$. Then f and $\widehat{f}$ are continuous and

$$(4.5) \qquad\qquad \sum_{n \in \mathbb{Z}^d} f(x+n) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{2\pi i n \cdot x}$$

for all $x \in \mathbb{R}^d$.

Applying to the function $f(x) = p(x,y)$ for which $\widehat{f}(n) = e^{-2\pi|n|y}$, we obtain that

$$(4.6) \qquad\qquad h(x,y) = \sum_{n \in \mathbb{Z}^d} p_0(x+n, y) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi|n|y} e^{2\pi i n \cdot x}.$$

From the fact that the Poisson kernel for the unit disc $\mathcal{P}_r(\theta)$ is given by

$$\mathcal{P}_r(\theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi,$$

we see that for $d = 1$,

$$(4.7) \qquad\qquad h(x,y) = \frac{1}{2\pi} \mathcal{P}_{e^{-2\pi y}}(2\pi x) = \frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)}.$$

This explicit expression for h when $d = 1$ is well known, see for example [42, pg.70]. It was derived in [10, Lemma 3.1] by a different argument and used there for many calculations. In particular, for $d = 1$, such a formula permits explicit calculations for various quantities involving the function $\frac{1}{h(x,y)}$, see [10]. In Section 8 we will use this in the calculation of the Fourier transform of the kernel $\mathbb{K}_\mathbb{H}$ for the probabilistic discrete Hilbert transform. For $d \geq 2$, while we can express $h(x,y)$ in various other forms besides (4.2) and (4.6), it does not seem possible to write such a convenient closed formula that will facilitate calculations with $\frac{1}{h(x,y)}$ in a similar manner.

4.2. h-**harmonic extension.** Let $f : \mathbb{Z}^d \to \mathbb{R}$ be a function of compact support, that is, $f(n) = 0$ all but finitely many $n \in \mathbb{Z}^d$. Define

$$(4.8) \qquad u_f(x, y) = \sum_{n \in \mathbb{Z}^d} f(n) \frac{p_n(x, y)}{h(x, y)}.$$

Note that u_f is h-harmonic. That is, $\Delta(h u_f) = 0$. Equivalently, $u_f(x, y)$ is harmonic in the upper half-space relative to the operator

$$\frac{1}{2} \Delta + \frac{\nabla h(x, y) \cdot \nabla}{h(x, y)}.$$

The following proposition provides information on the boundary values of u_f.

Proposition 4.3. *For each $x \in \mathbb{R}^d$, $p(x, y)/h(x, y)$ converges as $y \to 0$. Let $\Psi(x)$ be the limit, and $f \in \ell^p(\mathbb{Z}^d)$ be compactly supported. Define $f_{\text{ext}}(x) := \sum_{n \in \mathbb{Z}^d} f(n) \Psi(x - n)$. Then, $f_{\text{ext}}(n) = f(n)$ for all $n \in \mathbb{Z}^d$ and $\|f_{\text{ext}}\|_{L^p} \le \|f\|_{\ell^p}$.*

Proof. Suppose $x = 0$. Note that $p_n(0, y)/h(0, y) \le 1$ and

$$h(0, y) = p(0, y) + \sum_{n \ne 0} p(n, y) = p(0, y) + \sum_{n \ne 0} \frac{c_d y}{(|n|^2 + y^2)^{\frac{d+1}{2}}}$$

$$\le p(0, y) + c_d y \sum_{n \ne 0} |n|^{-(d+1)}.$$

Since the sum on the right hand side is finite, we have

$$\frac{1}{1 + C_d y^{d+1}} \le \frac{p(0, y)}{h(0, y)} \le 1.$$

Thus, if $x = 0$, the limit exists and $\Psi(0) = 1$.

Suppose $x = n \in \mathbb{Z}^d \setminus \{0\}$. Then, $\lim_{y \to 0} p(n, y) = 0$ and

$$\frac{1}{h(n, y)} \le \frac{1}{p(0, y)} = \frac{1}{c_d} y^d \to 0.$$

Thus, the limit exists and $\Psi(n) = 0$. For $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$, we have $\lim_{y \to 0} p(x, y)/y = c_d |x|^{-d-1}$ and

$$\lim_{y \to 0} \frac{h(x, y)}{y} = c_d \sum_{n \in \mathbb{Z}^d} |x - n|^{-(d+1)}.$$

Since the sum is finite for each $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$, we conclude that $\Psi(x)$ is well-defined. Note that

$$\int_{\mathbb{R}^d} \frac{p(x, y)}{h(x, y)} \, dx = \sum_{n \in \mathbb{Z}^d} \int_Q \frac{p_n(x, y)}{h(x, y)} \, dx = 1$$

and

$$\sum_{n \in \mathbb{Z}^d} \frac{p(x - n, y)}{h(x - n, y)} = \sum_{n \in \mathbb{Z}^d} \frac{p_n(x, y)}{h(x, y)} = 1.$$

By Fatou's lemma, we have

$$\int_{\mathbb{R}^d} \Psi(x)\, dx \le 1, \qquad \sum_{n \in \mathbb{Z}^d} \Psi(x-n) \le 1.$$

Thus, it follows from Hölder's inequality that

$$\|f_{\mathrm{ext}}\|_{L^p}^p = \int_{\mathbb{R}^d} |f_{\mathrm{ext}}(x)|^p\, dx \le \left(\sum_{n \in \mathbb{Z}^d} |f(n)|^p \int_{\mathbb{R}^d} \Psi(x-n)\, dx \right) \left(\sum_{n \in \mathbb{Z}^d} \Psi(x-n) \right)^{p-1} \le \|f\|_{\ell^p}^p.$$

$\square$

Remark 4.4. *Due to this Proposition, we call u_f the discrete harmonic extension of f.*

When $d = 1$, we can use (4.7) to compute that

$$\Psi_1(x) := \Psi(x) = \begin{cases} \frac{\sin^2(\pi x)}{\pi^2 x^2}, & x \ne 0, \\ 1, & x = 0. \end{cases}$$

For an arbitrary $d = 1, 2, \ldots$, set $\Psi_d(x) = \prod_{i=1}^d \Psi_1(x_i)$ and consider the function

$$\widetilde{f}_{\mathrm{ext}}(x) = \sum_{n \in \mathbb{Z}^d} f(n) \Psi_d(x - n).$$

It was proved in [52] that for all $d \ge 1$, there exists a dimensional constant $A_d \ge 1$ such that

(4.9) $$\frac{1}{A_d} \|f\|_{\ell^p} \le \|\widetilde{f}_{\mathrm{ext}}\|_{L^p} \le A_d \|f\|_{\ell^p}.$$

The compact support of the Fourier transform of $\Psi_d(x)$, which we are not able to verify in our case for $\Psi(x)$ when $d > 1$, is crucial for the first inequality in (4.9). The bounds in (4.9) were used in [52, Proposition 2.1] to show that the L^p-norm of a continuous Fourier multiplier operator T, when the multiplier is bounded and of compact support, controls the ℓ^p-norm of its discrete version T_{dis} with a constant depending on d. More precisely, suppose

$$\widehat{K}(\xi) = \int_{\mathbb{R}^d} K(x) e^{-2\pi i x \cdot \xi} dx$$

is bounded and supported on the fundamental cube $Q = [-\frac{1}{2}, \frac{1}{2})^d$. Define the Fourier multiplier $\widehat{Tf}(\xi) = \widehat{K}(\xi)\widehat{f}(\xi)$ and its discrete version by

$$T_{\mathrm{dis}}f(n) = \sum_{m \in \mathbb{Z}^d} K(m) f(n - m).$$

Fix $1 \le p \le \infty$. If T is bounded on $L^p(\mathbb{R}^d)$, then T_{dis} is bounded on $\ell^p(\mathbb{Z}^d)$ and

(4.10) $$\|T_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \le C_d \|T\|_{L^p \to L^p},$$

where $C_d = 3^d A_d^2$.

Remark 4.5. *The problem raised in [52, Remark (1), pg. 193],*

> *"it would be interesting to know if C_d can be taken to be independent of d, or for that matter if $C_d = 1$,"*

has been shown not to be the case, at least for p near 1. See [46] for details.

4.3. **Doob h-process and martingale transforms.** For the function $h(x, y)$ defined in (4.2), let Z_t be a solution of the stochastic differential equation

$$dZ_t = dB_t + \frac{\nabla h(Z_t)}{h(Z_t)} dt,$$

where $(B_t)_{t \geq 0}$ is the $(d+1)$-dimensional Brownian motion starting from $(x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}_+$. The lifetime of Z_t in the upper half-space is defined by $\tau = \inf\{t \geq 0 : Y_t = 0\}$. The lifetime τ is finite with probability one and the process Z_t only exits the upper half-space on $\mathbb{Z}^d \times \{0\}$. Indeed, at its lifetime Z_t approaches the point $n \in \mathbb{Z}^d$ with probability $\frac{p_n(x,y)}{h(x,y)}$, where (x, y) is the starting point of Z_t. For the basic properties and stochastic calculus for the Doob h-processes we refer the reader to [18, Chapter 3].

We denote by $\ell_c^p(\mathbb{Z}^d)$ the space of all compactly supported functions in $\ell^p(\mathbb{Z}^d)$. For $f \in \ell_c^p(\mathbb{Z}^d)$, we define

$$M_t = M_t^f = u_f(Z_t), \quad \text{for } t \in (0, \tau).$$

By Itô's formula, M_t^f is a martingale and satisfies

$$(4.11) \qquad M_t^f = M_0^f + \int_0^t \nabla u_f(Z_s) \cdot dZ_s + \frac{1}{2} \int_0^t \Delta u_f(Z_s) \, ds$$

$$= M_0^f + \int_0^t \nabla u_f(Z_s) \cdot dZ_s - \int_0^t \frac{\nabla h(Z_s) \cdot \nabla u_f(Z_s)}{h(Z_s)} \, ds$$

$$= M_0^f + \int_0^t \nabla u_f(Z_s) \cdot dB_s,$$

where $(B_t)_{t \geq 0}$ is the $(d+1)$-dimensional Brownian motion.

Let $\mathfrak{M}_{(d+1)}(\mathbb{R})$ be the space of all $(d+1) \times (d+1)$ real matrices and denote its norm by

$$\|A\| = \sup\{\|Av\| : v \in \mathbb{R}^{d+1}, \|v\| \leq 1\}.$$

By abuse of notation, for a matrix-valued function $A : \mathbb{R}^d \times \mathbb{R}_+ \to \mathfrak{M}_{(d+1)}(\mathbb{R})$, that is, $A(x, y) = (a_{ij}(x, y))_{1 \leq i, j \leq d+1}$ for all $x \in \mathbb{R}^d$ and $y > 0$, we define

$$(4.12) \qquad \|A\| = \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}_+} \|A(x, y)\|$$

$$= \sup\{\|A(x, y)v\| : v \in \mathbb{R}^{d+1}, \|v\| \leq 1, (x, y) \in \mathbb{R}^d \times \mathbb{R}_+\}.$$

We say a matrix $A(x, y) = (a_{ij}(x, y)) \in \mathfrak{M}_{(d+1)}(\mathbb{R})$ is *orthogonal* if $\langle A(x, y)v, v \rangle = \sum_{ij} a_{ij} v_i v_j = 0$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+$ and all $v \in \mathbb{R}^{d+1}$. Let $A(x, y)$ be a matrix-valued function on $\mathbb{R}^d \times \mathbb{R}_+$ and $f \in \ell_c^p(\mathbb{Z}^d)$. The martingale transform of $(M_t^f)_{t \geq 0}$ with respect to $A(x, y)$ is defined by

$$(A \star M^f)_t := \int_0^t A(Z_s) \nabla u_f(Z_s) \cdot dB_s$$

$$= \int_0^t A(Z_s) \nabla u_f(Z_s) \cdot dZ_s - \int_0^t \frac{A(Z_s) \nabla u_f(Z_s) \cdot \nabla h(Z_s)}{h(Z_s)} \, ds.$$

From the martingale inequalities in [21] (general A) and [16] (orthogonal A), respectively, we have the following

Theorem 4.6. *Let $1 < p < \infty$ and recall that $p^* = \max\{p, \frac{p}{p-1}\}$.*

(i) Let A be a matrix-valued function with $\|A\| < \infty$. Then we have

$$\|A \star M^f\|_p \leq (p^* - 1)\|A\|\|M^f\|_p.$$

(ii) If additionally A is orthogonal, then

$$\|A \star M^f\|_p \leq \cot\left(\frac{\pi}{2p^*}\right)\|A\|\|M^f\|_p.$$

5. DISCRETE OPERATORS ARISING FROM MARTINGALE TRANSFORMS

5.1. **The projection operators T_A and their ℓ^p boundedness.** For the rest of this paper we fix our starting point (x_0, y_0) to be $(0, w)$, $w > 0$. For $A : \mathbb{R}^d \times \mathbb{R}_+ \to \mathfrak{M}_{(d+1)}(\mathbb{R})$ with $\|A\| < \infty$, $f \in \ell_c^p(\mathbb{Z}^d)$ and $n \in \mathbb{Z}^d$, we define

$$(5.1) \qquad T_A^w(f)(n) = \mathrm{E}_{(0,w)}\Big[(A \star M^f)_\tau \Big| X_\tau = n\Big].$$

We call these operators *"projections of martingale transforms."* Our next goal is twofold. Firstly, we show that when $w \to \infty$, they give rise to a family of operators, denoted as T_A, which are bounded on $\ell_c^p(\mathbb{Z}^d)$, $1 < p < \infty$, with the same ℓ^p-bounds as those given in Theorem 4.6. In particular, these bounds are independent of d. Secondly, we compute their kernels. Although these constructions are in the style of Gundy–Varopoulos [38], we follow the approach in [3, Section 2] using the occupation time formula in terms of the Green's functions to compute their kernels.

For each $n \in \mathbb{Z}^d$, we consider the processes $(Z_t)_{t \geq 0}$ starting at $(0, w)$ and conditioned to exit the upper half-space at $(n, 0)$ and denote it by Z_t^n. Then Z_t^n is just Brownian in the upper half-space with drift $\frac{\nabla p_n}{p_n}$. Let us denote the Brownian motion which arises as the martingale part of Z_t^n by B_t^n, and the expectation of $(Z_t^n)_{t \geq 0}$ by $\mathrm{E}_{(0,w)}^n$. Then $T_A^w(f)$ can be written as

$$\begin{aligned}
T_A^w(f)(n) &= \mathrm{E}_{(0,w)}\Big[(A \star M^f)_\tau \Big| X_\tau = n\Big] \\
&= \mathrm{E}_{(0,w)}\Big[\int_0^\tau A(Z_s)\nabla u_f(Z_s) \cdot dZ_s - \int_0^\tau \frac{A(Z_s)\nabla u_f(Z_s) \cdot \nabla h(Z_s)}{h(Z_s)}ds \Big| X_\tau = n\Big] \\
&= \mathrm{E}_{(0,w)}^n\Big[\int_0^\tau A(Z_s^n)\nabla u_f(Z_s^n) \cdot dZ_s^n - \int_0^\tau \frac{A(Z_s^n)\nabla u_f(Z_s^n) \cdot \nabla h(Z_s^n)}{h(Z_s^n)}ds\Big] \\
&= \mathrm{E}_{(0,w)}^n\Big[\int_0^\tau A(Z_s^n)\nabla u_f(Z_s^n) \cdot dB_s^n + \int_0^\tau \frac{A(Z_s^n)\nabla u_f(Z_s^n) \cdot \nabla p_n(Z_s^n)}{p_n(Z_s^n)}ds \\
&\qquad - \int_0^\tau \frac{A(Z_s^n)\nabla u_f(Z_s^n) \cdot \nabla h(Z_s^n)}{h(Z_s^n)}ds\Big] \\
&= \mathrm{E}_{(0,w)}^n\Big[\int_0^\tau A(Z_s^n)\nabla u_f(Z_s^n) \cdot \Big(\frac{\nabla p_n(Z_s^n)}{p_n(Z_s^n)} - \frac{\nabla h(Z_s^n)}{h(Z_s^n)}\Big)ds\Big].
\end{aligned}$$

Next, we use the occupation time formula to write this expectation as an integral over $\mathbb{R}^d \times \mathbb{R}_+$. Let us denote the Green's function for the upper half-space with pole $(0, w)$

by $G_w(x, y)$. Then

$$G_w(x,y) = \begin{cases} \dfrac{1}{2\pi} \log\left(\dfrac{x^2 + (y+w)^2}{x^2 + (y-w)^2}\right), & d = 1, \\[4mm] \dfrac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}}\left(\dfrac{1}{(|x|^2 + |w-y|^2)^{\frac{d-1}{2}}} - \dfrac{1}{(|x|^2 + |w+y|^2)^{\frac{d-1}{2}}}\right), & d \geq 2. \end{cases}$$

Since the occupation time measure for the process Z_t^n is given by

$$\frac{p_n(x,y)G_w(x,y)}{p_n(0,w)},$$

it follows from the occupation time formula that

$$\begin{aligned} T_A^w(f)(n) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \frac{p_n(x,y)G_w(x,y)}{p_n(0,w)} A(x,y)\nabla u_f(x,y) \cdot \left(\frac{\nabla p_n(x,y)}{p_n(x,y)} - \frac{\nabla h(x,y)}{h(x,y)}\right) dy dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \frac{h(x,y)G_w(x,y)}{p_n(0,w)} A(x,y)\nabla u_f(x,y) \cdot \left(\frac{\nabla p_n(x,y)}{h(x,y)} - \frac{p_n(x,y)\nabla h(x,y)}{h(x,y)^2}\right) dy dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \frac{h(x,y)G_w(x,y)}{p_n(0,w)} A(x,y)\nabla u_f(x,y) \cdot \nabla\left(\frac{p_n(x,y)}{h(x,y)}\right) dy dx. \end{aligned}$$

For $m, n \in \mathbb{Z}^d$, we define the kernel for the operator T_A^w by

$$(5.2) \qquad \mathbb{K}_A^w(n,m) := T_A^w(\delta_m)(n)$$

where $\delta_m(n) = 1$ if $m = n$ and otherwise 0. Note that $u_{\delta_m}(x,y) = \frac{p_m(x,y)}{h(x,y)}$ and hence we have

$$(5.3) \qquad T_A^w(f)(n) = \sum_{m\in\mathbb{Z}^d} \mathbb{K}_A^w(n,m)f(m),$$

where

$$(5.4) \qquad \mathbb{K}_A^w(n,m) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \frac{G_w(x,y)}{p_n(0,w)} h(x,y)A(x,y)\nabla\left(\frac{p_m}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right) dy dx.$$

With the kernels $\mathbb{K}_A^w(n,m)$ defined for all $w > 0$, we like to compute the limit as $w \to \infty$ and study their properties. For each $m, n \in \mathbb{Z}^d$ and $f \in \ell_c^p(\mathbb{Z}^d)$, we define

$$(5.5) \qquad \mathbb{K}_A(n,m) = \lim_{w\to\infty} \mathbb{K}_A^w(n,m)$$

and

$$(5.6) \qquad T_A(f)(n) = \sum_{m\in\mathbb{Z}^d} \mathbb{K}_A(n,m)f(m).$$

The following theorem shows that T_A is well-defined and gives an explicit expression for it.

Theorem 5.1. *Let $A(x,y)$ be a matrix-valued function with $\|A\| < \infty$ and $m, n \in \mathbb{Z}^d$. Then, $\mathbb{K}_A^w(n,m)$ converges as $w \to \infty$ and*

$$(5.7) \quad \mathbb{K}_A(n,m) = \lim_{w\to\infty} \mathbb{K}_A^w(n,m) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} 2yh(x,y)A(x,y)\nabla\left(\frac{p_m}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right) dy dx.$$

Remark 5.2. *Note that if $A(x, y) = A$ is a constant matrix, then the change of variables $x = -z + m - n$ gives $K_A(n, m) = -K_{A^T}(n, m)$.*

Lemma 5.3. *Let $d \geq 1$, $n \in \mathbb{Z}^d$, and $(x, y) \in \mathbb{R}^d \times \mathbb{R}_+$. Then we have*

$$\lim_{w \to \infty} \frac{G_w(x, y)}{p_n(0, w)} = 2y.$$

Suppose that $d \geq 2$. Then

$$\frac{G_w(x, y)}{p_n(0, w)} \leq \begin{cases} C_d y & y \leq \frac{w}{2}, \\ C_d w (|\frac{x}{w}|^2 + (\frac{y}{w} - 1)^2)^{-(d-1)/2} & y > \frac{w}{2}. \end{cases}$$

when $n \in \mathbb{Z}^d$ and $w \geq |n|$.
    For $d = 1$, we have

$$\frac{G_w(x, y)}{p_n(0, w)} \leq 2yg\left(\frac{y}{w}\right)$$

if $n \in \mathbb{Z}^d$ and $w \geq |n|$, where

$$g(t) = \frac{1}{2t} \log(1 + 4t(t - 1)^{-2}).$$

Proof. The case $d = 1$ was proven in [10, Lemma 3.3]. For $d \geq 2$, we use the mean value theorem to get

$$\frac{1}{(|x|^2 + |w - y|^2)^{\frac{d-1}{2}}} - \frac{1}{(|x|^2 + |w + y|^2)^{\frac{d-1}{2}}} = 2(d - 1)wy(|x|^2 + |w - y|^2 + 4wy\varepsilon)^{-\frac{d+1}{2}}$$

$$= 2(d - 1)yw^{-d}(1 + |\tfrac{x}{w}|^2 + |\tfrac{y}{w}|^2 + 2(2\varepsilon - 1)\tfrac{y}{w})^{-\frac{d+1}{2}}$$

for some $\varepsilon \in (0, 1)$. The first part follows now from

$$\frac{G_w(x, y)}{p_n(0, w)} = \frac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d+1}{2}}} \frac{2(d - 1)yw^{-d}(1 + |\frac{x}{w}|^2 + |\frac{y}{w}|^2 + 2(2\varepsilon - 1)\frac{y}{w})^{-\frac{d+1}{2}}}{c_d w^{-d}(1 + \frac{|n|^2}{|w|^2})^{-\frac{d+1}{2}}}$$

$$= \frac{2y(1 + \frac{|n|^2}{|w|^2})^{\frac{d+1}{2}}}{(1 + |\frac{x}{w}|^2 + |\frac{y}{w}|^2 + 2(2\varepsilon - 1)\frac{y}{w})^{\frac{d+1}{2}}},$$

because the right-hand side clearly converges to $2y$ as $w \to \infty$. To prove the second part of the statement, we observe that

$$\frac{G_w(x, y)}{p_n(0, w)} = \frac{2y(1 + \frac{|n|^2}{|w|^2})^{\frac{d+1}{2}}}{(1 + |\frac{x}{w}|^2 + |\frac{y}{w}|^2 + 2(2\varepsilon - 1)\frac{y}{w})^{\frac{d+1}{2}}} \leq 2y\left(\frac{2}{|\frac{y}{w} - 1|^2}\right)^{\frac{d+1}{2}} \leq 2^{1+3(d+1)/2}y,$$

when $w \geq |n|$ and $y \leq \frac{w}{2}$. Furthermore, when $y > \frac{w}{2}$, we have

$$\frac{G_w(x, y)}{p_n(0, w)} \leq \frac{\Gamma(\frac{d-1}{2})}{2\pi^{(d+1)/2} c_d} \frac{(|n|^2 + w^2)^{(d+1)/2}}{w(|x|^2 + (y - w)^2)^{(d-1)/2}}$$

$$\leq \frac{1}{d - 1} \frac{(2w)^{d+1}}{w^d(|\frac{x}{w}|^2 + (\frac{y}{w} - 1)^2)^{(d-1)/2}}$$

when $w \geq |n|$.   □

Lemma 5.4. *Let $d \geq 1$, $n \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, and $y > 0$, then we have*

$$\frac{|\nabla p_n(x, y)|}{p_n(x, y)} \leq \frac{d}{y}, \quad \frac{|\nabla h(x, y)|}{h(x, y)} \leq \frac{d}{y}.$$

Proof. Direct computation gives

$$(5.8) \qquad \frac{\partial}{\partial x_i} p_n(x, y) = -\frac{c_d(d+1)(x_i - n_i)y}{(|x - n|^2 + y^2)^{\frac{d+3}{2}}} = -\frac{(d+1)(x_i - n_i)}{|x - n|^2 + y^2} p_n(x, y),$$

$$(5.9) \qquad \frac{\partial}{\partial y} p_n(x, y) = \frac{c_d(|x - n|^2 - dy^2)}{(|x - n|^2 + y^2)^{\frac{d+3}{2}}} = \frac{(|x - n|^2 - dy^2)}{y(|x - n|^2 + y^2)} p_n(x, y).$$

This in turn gives

$$\begin{aligned}
\frac{|\nabla p_n(x, y)|^2}{p_n(x, y)^2} &= \frac{1}{y^2(|x - n|^2 + y^2)^2}\left((d+1)^2 y^2 |x - n|^2 + (|x - n|^2 - dy^2)^2\right) \\
&= \frac{1}{y^2(|x - n|^2 + y^2)^2}\left(|x - n|^4 + (d^2 + 1)y^2|x - n|^2 + d^2 y^4\right) \\
&= \frac{1}{y^2(|x - n|^2 + y^2)^2}\left((|x - n|^2 + d^2 y^2)(|x - n|^2 + y^2)\right) \\
&= \frac{|x - n|^2 + d^2 y^2}{y^2(|x - n|^2 + y^2)} \\
&\leq \frac{d^2}{y^2}.
\end{aligned}$$

Let $N > 0$ and set $h^N(x, y) = \sum_{n \in \mathbb{Z}^d, |n| \leq N} p_n(x, y)$. Note that $h^N(x, y) \to h(x, y)$ and $\nabla h^N(x, y) \to \nabla h(x, y)$ as $N \to \infty$ uniformly on compact sets in $\mathbb{R}^d \times \mathbb{R}_+$. We then have

$$\begin{aligned}
|\nabla h^N(x, y)| &\leq \sum_{n \in \mathbb{Z}^d, |n| \leq N} |\nabla p_n(x, y)| \\
&\leq \frac{d}{y} \sum_{n \in \mathbb{Z}^d, |n| \leq N} p_n(x, y) \\
&\leq \frac{d}{y} h(x, y).
\end{aligned}$$

Letting $N \to \infty$, we get the desired result. $\qquad \square$

Proof of Theorem 5.1. Let

$$j_{n,m}(x, y, w) = \frac{G_w(x, y)}{p_n(0, w)} h(x, y) A(x, y) \nabla\left(\frac{p_m}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right),$$

then $\mathbb{K}_A^w(n, m) = \iint j_{n,m}(x, y, w)\, dxdy$. Let $w \geq |n|$ and define

$$\begin{aligned}
j_{n,m}^{(1)}(x, y, w) &= j_{n,m}(x, y, w)\mathbb{1}_{\{0 < y < w/2\}}, \\
j_{n,m}^{(2)}(x, y, w) &= j_{n,m}(x, y, w)\mathbb{1}_{\{y \geq w/2\}}.
\end{aligned}$$

We claim that

$$\lim_{w\to\infty} \iint j_{n,m}^{(1)}(x,y,w)\,dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} 2yh(x,y)A\nabla\left(\frac{p_m}{h}\right)\cdot\nabla\left(\frac{p_n}{h}\right)\,dydx,$$

$$\lim_{w\to\infty} \iint j_{n,m}^{(2)}(x,y,w)\,dxdy = 0.$$

By Lemma 5.4, we have

$$|\nabla(\log\frac{p_m(x,y)}{h(x,y)})| \le \frac{|\nabla p_m(x,y)|}{p_m(x,y)} + \frac{|\nabla h(x,y)|}{h(x,y)} \le \frac{2d}{y}$$

and

$$\left|A(x,y)\nabla\left(\log\frac{p_m(x,y)}{h(x,y)}\right)\cdot\nabla\left(\log\frac{p_n(x,y)}{h(x,y)}\right)\right| \le \frac{C_{d,A}}{y^2}.$$

Note that if $0 < y < w/2$, then $g(y/w) \le C$ for some $C > 0$. By Lemma 5.3,

$$G_w(x,y)/p_n(0,w) \le Cy.$$

Thus, we have

$$j_{n,m}^{(1)}(x,y,w) = \frac{G_w(x,y)}{p_n(0,w)}\frac{p_m p_n}{h}A\nabla\left(\log\frac{p_m}{h}\right)\cdot\nabla\left(\log\frac{p_n}{h}\right)$$

$$\le \frac{C_{d,A}p_m(x,y)p_n(x,y)}{yh(x,y)}.$$

If $y \ge 1$, then it follows from Lemma 4.1 that

$$\frac{p_m(x,y)p_n(x,y)}{h(x,y)} \le Cp_m(x,y)p_n(x,y) \le C_{d,m,n}\frac{y^2}{(|x|^2+y^2)^{d+1}}.$$

Since

$$\int_1^\infty \int_{\mathbb{R}^d} \frac{y}{(|x|^2+y^2)^{d+1}}\,dxdy < \infty,$$

it follows from the dominated convergence theorem that

$$\lim_{w\to\infty}\int_1^\infty \int_{\mathbb{R}^d} j_{n,m}^{(1)}(x,y,w)\,dxdy = \int_1^\infty\int_{\mathbb{R}^d} 2yh(x,y)A(x,y)\nabla\left(\frac{p_m}{h}\right)\cdot\nabla\left(\frac{p_n}{h}\right)\,dxdy.$$

Suppose $0 < y < 1$ and $n \ne m$. Using $p_m(x,y) + p_n(x,y) \le h(x,y)$, we get

$$\frac{p_m(x,y)p_n(x,y)}{h(x,y)} \le \frac{c_d y}{(|x-n|^2+y^2)^{\frac{d+1}{2}} + (|x-m|^2+y^2)^{\frac{d+1}{2}}}$$

$$\le \frac{2^{\frac{d-1}{2}}c_d y}{(|x-n|^2+|x-m|^2+2y^2)^{\frac{d+1}{2}}}$$

$$= \frac{c_d y}{2(|x-(\frac{n+m}{2})|^2+y^2+|\frac{n+m}{2}|^2)^{\frac{d+1}{2}}}$$

$$\le C_{d,n,m}\frac{y}{(|x|^2+1)^{\frac{d+1}{2}}}.$$

Since

$$\int_0^1 \int_{\mathbb{R}^d} \frac{1}{(|x|^2 + 1)^{\frac{d+1}{2}}} \, dxdy < \infty,$$

we obtain that

$$\lim_{w \to \infty} \int_0^1 \int_{\mathbb{R}^d} j_{n,m}^{(1)}(x, y, w) \, dxdy = \int_0^1 \int_{\mathbb{R}^d} 2yh(x,y)A(x,y)\nabla\left(\frac{p_m}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right) dxdy$$

when $n \neq m$.

Suppose $0 < y < 1$ and $n = m$. For $f = \delta_n$, it follows from Itô's formula that

$$M_t^f = u_f(Z_t) = u_f(0, w) + \int_0^t \nabla u_f(Z_s) \cdot dB_s = \frac{p_n(0, w)}{h(0, w)} + \int_0^t \nabla\left(\frac{p_n(Z_s)}{h(Z_s)}\right) \cdot dB_s.$$

Note that $\mathrm{E}_{(0,w)}[M_\tau^f | X_\tau = n] = 1$. Applying (5.4) with $A(x, y) = \mathrm{Id}$, we get

$$\mathrm{E}_{(0,w)}\left[\int_0^\tau \nabla\left(\frac{p_n(Z_s)}{h(Z_s)}\right) \cdot dB_s \,\bigg|\, X_\tau = n\right] = \int_0^\infty \int_{\mathbb{R}^d} \frac{G_w(x, y)}{p_n(0, w)} h(x, y) \left|\nabla\left(\frac{p_n}{h}\right)\right|^2 dxdy,$$

which leads to

(5.10) $$\int_0^\infty \int_{\mathbb{R}^d} \frac{G_w(x, y)}{p_n(0, w)} h(x, y) \left|\nabla\left(\frac{p_n}{h}\right)\right|^2 dxdy = 1 - \frac{p_n(0, w)}{h(0, w)}.$$

Fix $N > 0$. Note that it follows from the proof of Lemma 5.3 that there exists a constant C depending only on d such that for large w,

$$\frac{G_w(x, y)}{p_n(0, w)} \geq Cy$$

for all $0 < y < 1$ and $|x| \leq N$. Thus, we obtain

$$\int_0^1 \int_{|x| \leq N} yh(x, y) \left|\nabla\left(\frac{p_n}{h}\right)\right|^2 dxdy \leq C.$$

By Lemma 5.3 and the dominated convergence theorem, we get

$$\lim_{w \to \infty} \int_0^1 \int_{|x| \leq N} j_{n,n}^{(1)}(x, y, w) \, dxdy = \int_0^1 \int_{|x| \leq N} 2yh(x,y)A(x,y)\nabla\left(\frac{p_n}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right) dxdy.$$

Since $|j_{n,n}^{(1)}(\cdot, \cdot, w)|$ is integrable over $\mathbb{R}^d \times (0, 1)$, we have

$$\int_0^1 \int_{|x| \geq N} j_{n,n}^{(1)}(x, y, w) \, dxdy \to 0$$

as $N \to \infty$. Using the previous argument, we see that

$$\left|2yh(x,y)A(x,y)\nabla\left(\frac{p_n}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right)\right| \leq C\frac{p_n(x,y)^2}{yh(x,y)} \leq C\frac{p_n(x,y)}{y} = \frac{C}{(|x-n|^2 + y^2)^{(d+1)/2}}.$$

Since the integral of $(|x-n|^2 + y^2)^{-(d+1)/2}$ over $|x| \geq N$ converges to 0 as $N \to \infty$, we get

$$\lim_{w \to \infty} \int_0^1 \int_{\mathbb{R}^d} j_{n,n}^{(1)}(x, y, w) \, dxdy = \int_0^1 \int_{\mathbb{R}^d} 2yh(x,y)A(x,y)\nabla\left(\frac{p_n}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right) dxdy$$

as desired.

For the other integral, we have

$$
\begin{aligned}
j_{n,m}^{(2)}(x,y,w) &\leq C_{d,n,m,A} \frac{y^2}{(|x|^2+y^2)^{d+1}} \cdot \frac{C_d w}{(|\frac{x}{w}|^2+|\frac{y}{w}-1|^2)^{(d-1)/2}} \cdot \frac{1}{y^2} \\
&= C_{d,n,m,A} \frac{1}{w^{2d+1}(|\frac{x}{w}|^2+|\frac{y}{w}|^2)^{d+1}(|\frac{x}{w}|^2+|\frac{y}{w}-1|^2)^{(d-1)/2}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d} j_{n,m}^{(2)}(x,y,w)\,dxdy &\leq \frac{C_{d,n,m,A}}{w^{2d+1}} \int_{\frac{w}{2}}^\infty \int_{\mathbb{R}^d} \frac{1}{(|\frac{x}{w}|^2+|\frac{y}{w}|^2)^{d+1}(|\frac{x}{w}|^2+|\frac{y}{w}-1|^2)^{(d-1)/2}}\,dxdy \\
&= \frac{C_{d,n,m,A}}{w^d} \int_{\frac{1}{2}}^\infty \int_{\mathbb{R}^d} \frac{1}{(|v|^2+s^2)^{d+1}(|v|^2+|s-1|^2)^{(d-1)/2}}\,dvds
\end{aligned}
$$

and the right-hand side clearly converges to zero as $w \to \infty$. $\qquad\square$

Theorem 5.5. *Let $f \in \ell^p(\mathbb{Z}^d)$, $1 < p < \infty$, and $A(x,y)$ be a matrix-valued function with $\|A\| < \infty$. Then*

$$
\tag{5.11} \|T_A(f)\|_{\ell^p} \leq (p^*-1)\|A\|\|f\|_{\ell^p}.
$$

If in addition, $A(x,y)$ is orthogonal for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}_+$, then

$$
\tag{5.12} \|T_A(f)\|_{\ell^p} \leq \cot\left(\frac{\pi}{2p^*}\right)\|A\|\|f\|_{\ell^p}.
$$

Proof. Suppose $f : \mathbb{Z}^d \to \mathbb{R}$ is compactly supported. By the sharp martingale inequality (Theorem 4.6) and Jensen's inequality for conditional expectations, we have

$$
\tag{5.13}
\begin{aligned}
\sum_{n\in\mathbb{Z}^d} |T_A^w(f)(n)|^p \frac{p_n(0,w)}{h(0,w)} &= \mathrm{E}_{(0,w)}[|T_A^w(f)(X_\tau)|^p] \\
&= \mathrm{E}_{(0,w)}[|\mathrm{E}_{(0,w)}[(A*M^f)_\tau|X_\tau]|^p] \\
&\leq \mathrm{E}_{(0,w)}[\mathrm{E}_{(0,w)}[|(A*M^f)_\tau|^p|X_\tau]] \\
&= \mathrm{E}_{(0,w)}[|(A\star M^f)_\tau|^p] \\
&\leq (p^*-1)^p\|A\|^p \mathrm{E}_{(0,w)}[|M_\tau^f|^p] \\
&= (p^*-1)^p\|A\|^p \sum_{n\in\mathbb{Z}^d} |f(n)|^p \frac{p_n(0,w)}{h(0,w)}.
\end{aligned}
$$

Since

$$
p_n(0,w) = \frac{c_d w}{(|n|^2+w^2)^{\frac{d+1}{2}}} = \frac{c_d}{w^d} \frac{1}{\left(\frac{|n|^2}{w^2}+1\right)^{\frac{d+1}{2}}} \leq \frac{c_d}{w^d},
$$

we get

$$
\frac{1}{c_d} \sum_{n\in\mathbb{Z}^d} |T_A^w(f)(n)|^p w^d p_n(0,w) \leq (p^*-1)^p\|A\|^p \sum_{n\in\mathbb{Z}^d} |f(n)|^p.
$$

Recall that $T_A(f)(n)$ is the (pointwise) limit of $T_A^w(f)(n)$. By Fatou's lemma, we get

$$\sum_{n \in \mathbb{Z}^d} |T_A(f)(n)|^p \leq \liminf_{w \to \infty} \frac{1}{c_d} \sum_{n \in \mathbb{Z}^d} |T_A^w(f)(n)|^p w^d p_n(0, w)$$

$$\leq (p^* - 1)^p \|A\|^p \sum_{n \in \mathbb{Z}^d} |f(n)|^p,$$

which proves (5.11). The proof of (5.12) follows from the same argument using the second part of Theorem 4.6. □

The following Littlewood–Paley inequality is the analogue in our current setting of the inequalities in [5, Corollaries 3.42 and 3.9.2].

Corollary 5.6. *Let $p \in (1, \infty)$, $q = p/(p-1)$, $f \in \ell^p(\mathbb{Z}^d)$, and $g \in \ell^q(\mathbb{Z}^d)$, then*

$$\int_0^\infty \int_{\mathbb{R}^d} 2yh(x, y) |\nabla u_f(x, y)| |\nabla u_g(x, y)| \, dx \, dy \leq (p^* - 1) \|f\|_{\ell^p} \|g\|_{\ell^q}.$$

Proof. Let $A(x, y)$ be a matrix-valued function with $\|A\| < \infty$. Assume that f and g have compact supports. By (5.7) and Theorem 5.5, we have

$$\left| \sum_{n \in \mathbb{Z}^d} T_A f(n) g(n) \right| = \left| \sum_{n,m \in \mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}_+} 2yh(x, y) A(x, y) \nabla\left(\frac{p_m}{h}\right) \cdot \nabla\left(\frac{p_n}{h}\right) dy \, dx \right) f(m) g(n) \right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} 2yh(x, y) A(x, y) \nabla u_f(x, y) \cdot \nabla u_g(x, y) \, dy \, dx \right|$$

$$\leq (p^* - 1) \|A\| \|f\|_{\ell^p} \|g\|_{\ell^q}.$$

Define

$$A_{ij}(x, y) = \frac{\partial_j u_f(x, y) \partial_i u_g(x, y)}{|\nabla u_f(x, y)| |\nabla u_g(x, y)|}$$

for $1 \leq i, j \leq d + 1$. Here, we used the notations $\partial_i = \partial_{x_i}$ for $i = 1, 2, \cdots, d$ and $\partial_{d+1} = \partial_y$. Since $\|A\| \leq 1$ and $A \nabla u_f \nabla u_g = |\nabla u_f| |\nabla u_g|$, the proof is complete. □

Remark 5.7. *Let $a : \mathbb{Z}^d \to (0, \infty)$. We define $h_a(x, y) = \sum_{n \in \mathbb{Z}^d} a(n) p_n(x, y)$ and*

$$u_f^a(x, y) = \sum_{n \in \mathbb{Z}^d} f(n) \frac{a(n) p_n(x, y)}{h^a(x, y)}.$$

For $f : \mathbb{Z}^d \to \mathbb{R}$ we define its weighted $\ell^p(a)$ norm by

$$\|f\|_{\ell^p(a)} = \left(\sum_{n \in \mathbb{Z}^d} |f(n)|^p a(n) \right)^{1/p}.$$

Following the same argument with $h_a(x, y)$, and under the assumption that

$$\sum_{n \neq 0} a(n) |n|^{-d-1} < \infty,$$

which guarantees that the harmonic function $h_a(x, y)$ is well-defined, we can define a discrete operator $T_A^{h_a}$ associated with h_a and obtain the weighted norm inequalities:

$$(5.14) \qquad \|T_A^{h_a} f\|_{\ell^p(a)} \leq (p^* - 1)\|A\| \|f\|_{\ell^p(a)},$$

for any A. If in addition A is orthogonal then,

$$(5.15) \qquad \|T_A^{h_a} f\|_{\ell^p(a)} \leq \cot\left(\frac{\pi}{2p^*}\right) \|A\| \|f\|_{\ell^p(a)}.$$

6. Discrete Calderón–Zygmund operators

6.1. **Discrete Calderón–Zygmund operators and norm estimates.** Let T be an operator acting on the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. We say T is a Calderón–Zygmund operator if it is bounded in L^2 and can be written as

$$(6.1) \qquad Tf(x) = p.v. \int_{\mathbb{R}^d} K(x, z) f(z) \, dz$$

where K is continuously differentiable off the diagonal with the bounds

$$(6.2) \quad |K(x, z)| \leq \frac{\kappa}{|x - z|^d}, \quad |\nabla_x K(x, z)| \leq \frac{\kappa}{|x - z|^{d+1}}, \quad |\nabla_z K(x, z)| \leq \frac{\kappa}{|x - z|^{d+1}},$$

for $x \neq z$, for some universal constant κ.

The Calderón–Zygmund operator T as above are bounded in L^p, for $1 < p < \infty$ (see [37, Chapter 8]). Here we will consider Calderón–Zygmund operators which are of convolution type. That is, their kernels are of the form $K(x, z) = K(x - z)$ satisfying

$$(6.3) \qquad K \in C^1(\mathbb{R}^d \setminus \{0\}), \quad |K(z)| \leq \kappa |z|^{-d}, \quad |\nabla K(z)| \leq \kappa |z|^{-(d+1)},$$

for some universal constant κ.

For these operators, Calderón and Zygmund [22] defined their discrete analogues by

$$(6.4) \qquad T_{\text{dis}}(f)(n) = \sum_{m \in \mathbb{Z}^d \setminus \{n\}} K(n - m) f(m), \quad f \in \ell^p(\mathbb{Z}^d).$$

As already mentioned in the introduction, M. Riesz [64] showed that in dimension 1, the boundedness of H on $L^p(\mathbb{R})$ implies the boundedness of H_{dis} on $\ell^p(\mathbb{Z})$. In the "Added in proof" section of their paper Calderón and Zygmund observed that the boundedness of T on $L^p(\mathbb{R}^d)$ leads to the boundedness of T_{dis} on $\ell^p(\mathbb{Z}^d)$. In fact, Calderón and Zygmund simply remarked ([22, pg. 138]) that "for $n = 1$ this remark is due to M. Riesz, and the proof in the case of general n follows a similar pattern" (here their $n = d$) and no further details are provided. For the sake of completeness and because we wish to keep track of constants, we provide the proof here. Recall that the truncated operator T_ε is defined by $T_\varepsilon(f) = K_\varepsilon * f$ where $K_\varepsilon(x) = K(x) \mathbb{1}_{\{|x| \geq \varepsilon\}}$, T_ε satisfies $\|T_\varepsilon f\|_{L^p} \leq C_p \|f\|_{L^p}$, where C_p is independent of ε and $\lim_{\varepsilon \to 0} T_\varepsilon f$ exists in $L^p(\mathbb{R}^d)$ and a.e. We denote the limit operator by T.

Proposition 6.1. *Let T be the Calderón–Zygmund operator with convolution kernel $K(x)$ satisfying (6.3). Then, T_{dis} is bounded on $\ell^p(\mathbb{Z}^d)$, $1 < p < \infty$. Furthermore, we have*

$$\|T_{\text{dis}}\|_{\ell^p \to \ell^p} \leq \|T_1\|_{L^p \to L^p} + C(d, \kappa).$$

Proof. As asserted by Calderón and Zygmund, the proof follows the argument of Riesz. Let $p \in (1, \infty)$ and q be the conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in \ell^p(\mathbb{Z}^d)$ and $g \in \ell^q(\mathbb{Z}^d)$. Define $F \in L^p(\mathbb{R}^d)$ and $G \in L^q(\mathbb{R}^d)$ by $F(x) = f(n)$ and $G(x) = g(n)$ for $x \in n + Q$, $Q = [-\frac{1}{2}, \frac{1}{2})^d$. Then,

$$\int_{\mathbb{R}^d} T_1(F)(x)G(x)\,dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_1(y-x)F(x)G(y)\,dxdy$$

$$= \sum_{n,m\in\mathbb{Z}^d} \left(\int_{m+Q} \int_{n+Q} K_1(y-x)\,dxdy \right) f(n)g(m)$$

$$= \sum_{n\in\mathbb{Z}^d} T_{\mathrm{dis}}(f)(n)g(n) + \sum_{n,m\in\mathbb{Z}^d} \widetilde{K_1}(n-m)f(n)g(m),$$

where

$$\widetilde{K_1}(n-m) = \int_{m+Q} \int_{n+Q} K_1(y-x)\,dxdy - K_1(m-n)$$

$$= \int_Q \int_Q (K_1(m-n+s-t) - K_1(m-n))\,dtds.$$

Using $|\nabla K_1(x-z)| \le \kappa |x-z|^{-(d+1)}$, we have

$$|\widetilde{K_1}(n-m)| \le C(d,\kappa)|n-m|^{-(d+1)},$$

for $|m-n|$ large enough. Since $|m|^{-(d+1)}$ is summable, we have

$$\left| \sum_{n,m\in\mathbb{Z}^d} \widetilde{K_1}(n,m)f(n)g(m) \right| \le \left(\sum_{n,m\in\mathbb{Z}^d} |\widetilde{K_1}(n,m)||f(n)|^p \right)^{1/p} \left(\sum_{n,m\in\mathbb{Z}^d} |\widetilde{K_1}(n,m)||g(m)|^q \right)^{1/q}$$

$$\le C(d,\kappa)\|f\|_{\ell^p}\|g\|_{\ell^q}$$

and this gives

$$\|T_{\mathrm{dis}}\|_{\ell^p\to\ell^p} \le \|T_1\|_{L^p\to L^p} + C(d,\kappa),$$

where the constant $C(d,\kappa)$ depends on d and κ but not on p. $\qquad\square$

Riesz's argument was modified in [41] to prove a discrete A_p-weighted version of the celebrated Hunt-Muckenhoupt-Wheeden weighted norm inequality for the Hilbert transform. See also [47, Theorem 4.6].

In [69], Titchmarsh gave (with a slightly different version of H) a different proof of Riesz's theorem by first showing that H_{dis} is bounded on ℓ^p and from this that H is bounded in L^p and that in fact $\|H\|_{L^p\to L^p} \le \|H_{\mathrm{dis}}\|_{\ell^p\to\ell^p}$. We show next that a similar result holds for singular integrals that commute with dilations. More precisely, consider singular integrals with kernels of the form $K(x) = \frac{\Omega(x)}{|x|^d}$, where Ω is homogeneous of degree zero; $\Omega(rx) = \Omega(x)$, for all $r > 0$. We assume that Ω satisfies the necessary hypothesis (see for example [65, Theorem 3]) so that the singular integral is bounded on L^p. That is, (i) Ω is bounded, (ii) Dini continuous, and (iii) its integral on the sphere is 0.

Theorem 6.2. *Suppose $K(x) = \frac{\Omega(x)}{|x|^d}$ satisfying (i)–(iii) and $\Omega(-x) = -\Omega(x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $1 < p < \infty$, we have $\|T\|_{L^p \to L^p} \leq \|T_{\mathrm{dis}}\|_{\ell^p \to \ell^p}$.*

Proof. We define the continuous-discrete operator $\widetilde{T}_{\mathrm{dis}}$ on $L^p(\mathbb{R}^d)$ by

$$\widetilde{T}_{\mathrm{dis}}(F)(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(n) F(x - n), \quad F \in L^p, \, x \in \mathbb{R}^d.$$

Let $Q = [-\frac{1}{2}, \frac{1}{2})^d$. For $f \in \ell^p(\mathbb{Z}^d)$, let $F(x) = \sum_{n \in \mathbb{Z}^d} f(n) \mathbb{1}_Q(x - n)$. Then, $\|F\|_{L^p} = \|f\|_{\ell^p}$ and

$$\widetilde{T}_{\mathrm{dis}}(F)(n) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) \sum_{l \in \mathbb{Z}^d} f(l) \mathbb{1}_Q(n - m - l)$$

$$= \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) f(n - m)$$

$$= T_{\mathrm{dis}}(f)(n),$$

which implies $\|T_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \leq \|\widetilde{T}_{\mathrm{dis}}\|_{L^p \to L^p}$. On the other hand, for any $F \in L^p$ we have,

$$\|\widetilde{T}_{\mathrm{dis}} F\|_{L^p}^p = \int_{\mathbb{R}^d} \Big| \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) F(x - m) \Big|^p dx$$

$$= \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \int_Q \Big| \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) F(x + n - m) \Big|^p dx$$

$$\leq \|T_{\mathrm{dis}}\|_{\ell^p \to \ell^p}^p \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \int_Q |F(x + n)|^p dx$$

$$= \|T_{\mathrm{dis}}\|_{\ell^p \to \ell^p}^p \|F\|_{L^p}^p.$$

Thus in fact, $\|\widetilde{T}_{\mathrm{dis}}\|_{L^p \to L^p} = \|T_{\mathrm{dis}}\|_{\ell^p \to \ell^p}$.

Let $\varepsilon > 0$ and define $\tau_\varepsilon F(x) := \varepsilon^{\frac{d}{p}} F(\varepsilon x)$. Then $\|\tau_\varepsilon F\|_p = \|F\|_p$, for $F \in L^p(\mathbb{R}^d)$. Let $\widetilde{T}_{\mathrm{dis}}^\varepsilon F(x) = \tau_{\frac{1}{\varepsilon}} \widetilde{T}_{\mathrm{dis}} \tau_\varepsilon F(x)$. Now suppose F is smooth with compact support. Then

$$\widetilde{T}_{\mathrm{dis}}^\varepsilon F(x) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m) F(x - \varepsilon m)$$

$$= \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \frac{\Omega(\varepsilon m)}{|\varepsilon m|^d} F(x - \varepsilon m) \varepsilon^d$$

$$= \frac{\varepsilon^d}{2} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \frac{\Omega(\varepsilon m)}{|\varepsilon m|^d} (F(x - \varepsilon m) - F(x + \varepsilon m)).$$

For each $r > 0$, we have

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^d}{2} \sum_{\substack{m \in \mathbb{Z}^d \setminus \{0\}, \\ |\varepsilon m| > r}} \frac{\Omega(\varepsilon m)}{|\varepsilon m|^d} (F(x - \varepsilon m) - F(x + \varepsilon m)) = \frac{1}{2} \int_{|y| > r} \frac{\Omega(y)}{|y|^d} (F(x - y) - F(x + y)) \, dy.$$

On the other hand, since Ω is bounded and F is smooth on compact support, it follows that

$$\left| \frac{\varepsilon^d}{2} \sum_{\substack{m \in \mathbb{Z}^d \setminus \{0\}, \\ |\varepsilon m| \leq r}} \frac{\Omega(\varepsilon m)}{|\varepsilon m|^d} (F(x - \varepsilon m) - F(x + \varepsilon m)) \right| \leq C \varepsilon^d \sum_{\substack{m \in \mathbb{Z}^d \setminus \{0\}, \\ |\varepsilon m| \leq r}} \frac{1}{|\varepsilon m|^{d-1}} = Cr.$$

Similarly,

$$\left| \int_{|y| \leq r} \frac{\Omega(y)}{|y|^d} (F(x - y) - F(x + y)) \, dy \right| \leq C \int_{|y| \leq r} |y|^{1-d} \, dy = Cr.$$

Therefore, we get

(6.5)
$$\lim_{\varepsilon \to 0} \widetilde{T}_{\mathrm{dis}}^{\varepsilon} F(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon^d}{2} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \frac{\Omega(\varepsilon m)}{|\varepsilon m|^d} (F(x - \varepsilon m) - F(x + \varepsilon m))$$

$$= \frac{1}{2} \lim_{r \to 0} \int_{|y| > r} \frac{\Omega(y)}{|y|^d} (F(x - y) - F(x + y)) \, dy$$

$$= T(F)(x).$$

By Fatou's lemma, we get

$$\|T(F)\|_{L^p} \leq \liminf_{\varepsilon \downarrow 0} \|\widetilde{T}_{\mathrm{dis}}^{\varepsilon}(F)\|_{L^p} \leq \|\widetilde{T}_{\mathrm{dis}}(F)\|_{L^p},$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The "continuous-discrete operator" versions have been used in several places to bound the norm of the continuous version by that of its discrete versions, see for example [47, 58].

6.2. **A conjecture on the ℓ^p-norms of the discrete Riesz transforms.** The canonical examples of Calderón–Zygmund operators that satisfy the assumptions of both Proposition 6.1 and Theorem 6.2 are the classical Riesz transforms on $\mathbb{R}^d$ already defined in (1.12).

The Riesz transforms arise naturally from the Poisson semigroup and its connection to the Laplacian. That is, if we let $P_y f(x)$ be the convolution of the function f with the Poisson kernel $p(\cdot, y)$ as in (4.1), then in fact,

(6.6)
$$R^{(k)} f(x) = \int_0^\infty \frac{\partial}{\partial x_k} P_y f(x) \, dy = \frac{\partial}{\partial x_k} (-\Delta)^{-1/2} f(x).$$

With this interpretation, the Riesz transforms can be defined in a variety of analytic and geometric settings, including manifolds, Lie groups, and Wiener space. We briefly recall here the Gundy–Varopoulos [38] representation of $R^{(k)}$, referring the reader to [3, 5] for details and applications. Let B_t be the standard Brownian motion in the upper half-space $\mathbb{R}_+^{d+1}$ starting at the point $(0, y)$ and τ its exit time. Consider the conditional expectations operators

(6.7)
$$\mathrm{E}_{(0,y)} \left[\int_0^\tau \mathbb{H}^{(k)} \nabla U_f(B_s) \cdot dB_s \, \Big| \, B_\tau = x \right],$$

where $\mathbb{H}^{(k)}$ are the matrices in (1.19), $U_f(x,y) = P_y f(x)$. Just as we do in Section 5.1 for the discrete case, under the assumption that f is sufficiently smooth, the quantity in (6.7) converges pointwise to $R^{(k)}f(x)$, as $y \to \infty$.

We remark here that verifying the convergence of (6.7) to the Riesz transforms is much simpler than the corresponding convergence results in Section 5.1; see for example [5, p. 417]. It is also worth mentioning here that the original Gundy–Varopoulos paper used the so called "background radiation" process in the construction. That the background radiation is not needed was shown in [3].

It is proved in Laeng [48] that the L^p-norm of the truncated Hilbert transform H_ε

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{|z|>\varepsilon} \frac{f(x-z)}{z}\, dz,$$

coincides with that of the Hilbert transform H. That is, $\|H_\varepsilon\|_{p\to p} = \cot\left(\frac{\pi}{2p^*}\right)$ for every $\varepsilon > 0$.

The methods of rotations can be used to obtain the same for the truncated Riesz transforms. That is, let $K_\varepsilon^{(k)}(z) = K^{(k)}(z)\mathbb{1}_{\{|z|>\varepsilon\}}$ and $R_\varepsilon^{(k)}f(x) = K_\varepsilon^{(k)} * f(x)$ be the truncated Riesz transform. Then,

$$(6.8) \qquad\qquad \|R_\varepsilon^{(k)}\|_{L^p \to L^p} = \cot\left(\frac{\pi}{2p^*}\right),$$

for all $\varepsilon > 0$. Combining (1.15) with Fatou's Lemma, it suffices to show the upper inequality

$$(6.9) \qquad\qquad \|R_\varepsilon^{(k)}\|_{L^p \to L^p} \le \cot\left(\frac{\pi}{2p^*}\right).$$

This follows from the method of rotations ([37, Equation (4.2.17)]) applied with $\Omega(y) = c_d \frac{y_k}{|y|}$ and the additional observation that

$$\int_{\mathbb{S}^{d-1}} |\Omega(\theta)|\, d\theta = c_d \int_{\mathbb{S}^{d-1}} |\theta_k|\, d\theta = c_d \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} = \frac{2}{\pi}.$$

It follows from this and Propositions 6.1 and Theorem 6.2 that (1.16) holds. That is, we have the following

Corollary 6.3. *For $1 < p < \infty$, $k = 1, 2, \ldots, d$, we have*

$$\cot\left(\frac{\pi}{2p^*}\right) \le \|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p} \le \cot\left(\frac{\pi}{2p^*}\right) + C_d,$$

where C_d is a dimensional constant.

Conjecture 6.4. *For all $d > 1$, $1 < p < \infty$, $k = 1, \ldots, d$,*

$$(6.10) \qquad\qquad \|R_{\mathrm{dis}}^{(k)}\|_{\ell^p \to \ell^p} = \cot\left(\frac{\pi}{2p^*}\right).$$

A weaker but also interesting result would be

Problem 6.5. *Show that*

$$(6.11) \qquad\qquad \|R_{\mathrm{dis}}^{(k)}\|_{\ell^p} \le C_p,$$

where C_p is independent of d. Even more, it would be interesting to show that the vector of discrete Riesz transforms also has a bound independent of the dimension, as it was first proved by Stein in [66] *for the continuous version. That is, if*

$$(6.12) \qquad R_{\mathrm{dis}} f(n) = \left(\sum_{k=1}^{d} |R_{\mathrm{dis}}^{(k)} f(n)|^2 \right)^{1/2},$$

then

$$(6.13) \qquad \|R_{\mathrm{dis}} f\|_{\ell^p} \leq C_p \|f\|_{\ell^p},$$

where C_p is independent of d.

From the representation of the kernel $\mathbb{K}_A(n,m)$ in (5.7) and Remark 5.2, we have $T_{\mathbb{H}^{(k)}} = 2 T_{A^{(k)}}$ where $A_{ij}^{(k)} = 1$ for $i = d+1$, $j = k$ and otherwise 0. By [16, Theorem 4.2], we obtain that

$$\left\| \left(\sum_{k=1}^{d} |T_{\mathbb{H}^{(k)}} f|^2 \right)^{1/2} \right\|_{\ell^p} \leq 2(p^* - 1) \|f\|_{\ell^p}.$$

As for the best constant (6.13) for the vector, this is a well known problem even in the continuous case. The original proof with bound independent of dimension in [66], and different subsequent proofs shortly after, did not even give the sharp asymptotic behavior which is $\mathcal{O}(1/p - 1)$, as $p \to 1$ and $\mathcal{O}(p)$, as $p \to \infty$. The first proof providing this behavior was given in [4] with subsequent improvements with explicit constants in [16, Corollary 4.21] and [44, Theorem 5.1].

These problem are also motivated by the remark in [52, pg. 193] already discussed in connection to inequality (4.10).

7. Discrete Riesz transforms and their probabilistic counterparts

7.1. **Probabilistic Discrete Riesz Transforms and their norms.** The proof of the Conjecture 6.4 for $d = 1$ in [10] rests on the probabilistic construction of the operators in Section 4.3 for $d = 1$ applied to the operator $T_{\mathbb{H}}^w$ as in (5.12) with the matrix

$$(7.1) \qquad \mathbb{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which is orthogonal and of norm 1. Motivated by this and the Gundy–Varopoulos [38] probabilistic representation of the Riesz transforms $R^{(k)}$ on $\mathbb{R}^d$, $d \geq 1$ as in (6.7) and its many variants studied over the years (see for example [5, 6] and the many references therein), we consider the operators $T_{\mathbb{H}^{(k)}}$, where for each $k = 1, 2, \cdots, d$, the $(d+1) \times (d+1)$ matrix $\mathbb{H}^{(k)}$ is given by (1.19). As already noted $\mathbb{H}^{(k)}$ is orthogonal, $\|\mathbb{H}^{(k)}\| = 1$. This gives the operators $T_{\mathbb{H}^{(k)}}$, $k = 1, \ldots d$, which we call the *"probabilistic discrete Riesz transforms"*. By Theorem 5.1, their kernels are given by

$$(7.2) \qquad \mathbb{K}_{\mathbb{H}^{(k)}}(n,m) = \int_{\mathbb{R}^d} \int_0^\infty 2y h(x,y) \mathbb{H}^{(k)} \nabla \left(\frac{p_m(x,y)}{h(x,y)} \right) \cdot \nabla \left(\frac{p_n(x,y)}{h(x,y)} \right) \, dy dx.$$

Using the fact that $h(x + m, y) = h(x, y)$ for all $m \in \mathbb{Z}^d$, a change of variables shows that

(7.3)

$$\mathbb{K}_{\mathbb{H}^{(k)}}(n, m) = \mathbb{K}_{\mathbb{H}^{(k)}}(n-m) = \int_{\mathbb{R}^d} \int_0^\infty 2yh(x, y)\mathbb{H}^{(k)}\nabla\left(\frac{p_0(x, y)}{h(x, y)}\right)\cdot\nabla\left(\frac{p_{n-m}(x, y)}{h(x, y)}\right) dydx.$$

Note that when $d = 1$ the matrices in (1.19) reduce to the matrix $\mathbb{H}$ in (7.1) which gives the probabilistic discrete Hilbert transform $T_{\mathbb{H}}$.

The kernels $\mathbb{K}_{\mathbb{H}^{(k)}}$ enjoy the following properties that will be used below in several computations. For $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$, define

$$\widetilde{n} = (n_1, n_2, \cdots, -n_k, \ldots n_d).$$

It follows from (7.3) (or from (7.6) below) that $\mathbb{K}_{\mathbb{H}^{(k)}}(n) = -\mathbb{K}_{\mathbb{H}^{(k)}}(\widetilde{n})$. Thus we have $\mathbb{K}_{\mathbb{H}^{(k)}}(n) = 0$, if $n_k = 0$, and in particular, $\mathbb{K}_{\mathbb{H}^{(k)}}(0) = 0$. It also follows that $\mathbb{K}_{\mathbb{H}^{(k)}}(n) = -\mathbb{K}_{\mathbb{H}^{(k)}}(-n)$. These properties of $\mathbb{K}_{\mathbb{H}^{(k)}}$ will be used below in several computations. Because $\mathbb{K}_{\mathbb{H}^{(k)}}(0) = 0$ we often write the sum in the convolution over $\mathbb{Z}$ and not over $\mathbb{Z} \setminus \{0\}$.

By (5.12) of Theorem 5.5 and (5.7) of Theorem 5.1, we obtain the following

Theorem 7.1. *Suppose* $f \in \ell^p(\mathbb{Z}^d)$, $1 < p < \infty$. *Set*

(7.4)
$$T_{\mathbb{H}^{(k)}}(f)(n) = \sum_{m \in \mathbb{Z}^d} \mathbb{K}_{\mathbb{H}^{(k)}}(n - m)f(m).$$

Then,

(7.5)
$$\|T_{\mathbb{H}^{(k)}}f\|_{\ell^p} \leq \cot(\frac{\pi}{2p^*})\|f\|_{\ell^p}, \quad k = 1, 2, \ldots, d.$$

In the following proposition, we derive a different integral representation for the kernel $\mathbb{K}_{\mathbb{H}^{(k)}}(n)$ which will provide a relationship between the operators $T_{\mathbb{H}^{(k)}}$ and the CZ discrete Riesz transforms $R_{\mathrm{dis}}^{(k)}$. As we shall see, this representation allows us to prove that the ℓ^p-bound in (7.5) is best possible; see Theorem 7.5.

Theorem 7.2. *We have*

(7.6)
$$\mathbb{K}_{\mathbb{H}^{(k)}}(n) = \left(\int_{\mathbb{R}^d} \int_0^\infty \frac{U_n(x, y)}{h(x, y)} dydx\right) \mathbb{1}_{\{|n| \geq 1\}}(n)$$

$$= \left(4\int_{\mathbb{R}^d} \int_0^\infty \frac{S_n(x, y)}{h(x, y)} dydx - 3\int_{\mathbb{R}^d} \int_0^\infty \frac{T_n(x, y)}{h(x, y)} dydx\right) \mathbb{1}_{\{|n| \geq 1\}}(n)$$

(7.7)
$$= -4\int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h}\frac{\partial p_0}{\partial x_k}\frac{\partial}{\partial y}(yp_n) dydx$$

where

$$S_n(x, y) = \frac{2c_d^2(d + 1)x_k y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}}(|x - n|^2 + y^2)^{\frac{d+1}{2}}},$$

$$T_n(x, y) = \frac{4c_d^2(d + 1)^2 x_k y^4}{3(|x|^2 + y^2)^{\frac{d+3}{2}}(|x - n|^2 + y^2)^{\frac{d+3}{2}}},$$

and $U_n(x, y) = 4S_n(x, y) - 3T_n(x, y)$.

Proof. By (7.3), we have

$$
\mathbb{K}_{\mathbb{H}^{(k)}}(n)
$$
$$
= \int_{\mathbb{R}^d} \int_0^\infty 2yh \mathbb{H}^{(k)} \nabla \left(\frac{p_0}{h} \right) \cdot \nabla \left(\frac{p_n}{h} \right) \, dy dx
$$
$$
= \int_{\mathbb{R}^d} \int_0^\infty 2yh \left(\frac{\partial}{\partial x_k} \left(\frac{p_0}{h} \right) \frac{\partial}{\partial y} \left(\frac{p_n}{h} \right) - \frac{\partial}{\partial y} \left(\frac{p_0}{h} \right) \frac{\partial}{\partial x_k} \left(\frac{p_n}{h} \right) \right) \, dy dx
$$
$$
= \int_{\mathbb{R}^d} \int_0^\infty 2yh \left(\frac{1}{h^2} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} - \frac{p_0}{h^3} \frac{\partial h}{\partial x_k} \frac{\partial p_n}{\partial y} - \frac{p_n}{h^3} \frac{\partial p_0}{\partial x_k} \frac{\partial h}{\partial y} + \frac{p_0 p_n}{h^4} \frac{\partial h}{\partial x_k} \frac{\partial h}{\partial y} \right) \, dy dx
$$
$$
- \int_{\mathbb{R}^d} \int_0^\infty 2yh \left(\frac{1}{h^2} \frac{\partial p_0}{\partial y} \frac{\partial p_n}{\partial x_k} - \frac{p_0}{h^3} \frac{\partial h}{\partial y} \frac{\partial p_n}{\partial x_k} - \frac{p_n}{h^3} \frac{\partial p_0}{\partial y} \frac{\partial h}{\partial x_k} + \frac{p_0 p_n}{h^4} \frac{\partial h}{\partial y} \frac{\partial h}{\partial x_k} \right) \, dy dx
$$
$$
= \int_{\mathbb{R}^d} \int_0^\infty 2yh \left(\frac{1}{h^2} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} - \frac{p_0}{h^3} \frac{\partial h}{\partial x_k} \frac{\partial p_n}{\partial y} + \frac{p_0}{h^3} \frac{\partial h}{\partial y} \frac{\partial p_n}{\partial x_k} \right) \, dy dx
$$
$$
- \int_{\mathbb{R}^d} \int_0^\infty 2yh \left(\frac{1}{h^2} \frac{\partial p_0}{\partial y} \frac{\partial p_n}{\partial x_k} - \frac{p_n}{h^3} \frac{\partial p_0}{\partial y} \frac{\partial h}{\partial x_k} + \frac{p_n}{h^3} \frac{\partial p_0}{\partial x_k} \frac{\partial h}{\partial y} \right) \, dy dx.
$$

By the change of variable $x = n - x'$, we simplify

$$
\mathbb{K}_{\mathbb{H}^{(k)}}(n) = 4 \int_{\mathbb{R}^d} \int_0^\infty \left(\frac{y}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} + \frac{y p_0}{h^2} \frac{\partial h}{\partial y} \frac{\partial p_n}{\partial x_k} - \frac{y p_0}{h^2} \frac{\partial h}{\partial x_k} \frac{\partial p_n}{\partial y} \right) \, dy dx.
$$

It follows from integration by parts that

$$
\int_0^\infty \frac{y p_0}{h^2} \frac{\partial h}{\partial y} \frac{\partial p_n}{\partial x_k} \, dy = - \int_0^\infty h \frac{\partial}{\partial y} \left(\frac{y p_0}{h^2} \frac{\partial p_n}{\partial x_k} \right) \, dy
$$
$$
= - \int_0^\infty \left(\frac{p_0}{h} \frac{\partial p_n}{\partial x_k} + \frac{y}{h} \frac{\partial p_0}{\partial y} \frac{\partial p_n}{\partial x_k} - \frac{2 y p_0}{h^2} \frac{\partial h}{\partial y} \frac{\partial p_n}{\partial x_k} + \frac{y p_0}{h} \frac{\partial^2 p_n}{\partial y \partial x_k} \right) \, dy
$$
$$
= \int_0^\infty \left(\frac{p_0}{h} \frac{\partial p_n}{\partial x_k} + \frac{y}{h} \frac{\partial p_0}{\partial y} \frac{\partial p_n}{\partial x_k} + \frac{y p_0}{h} \frac{\partial^2 p_n}{\partial y \partial x_k} \right) \, dy.
$$

Similarly,

$$
- \int_{\mathbb{R}} \frac{y p_0}{h^2} \frac{\partial h}{\partial x_k} \frac{\partial p_n}{\partial y} \, dx_k = \int_{\mathbb{R}} h \frac{\partial}{\partial x_k} \left(\frac{y p_0}{h^2} \frac{\partial p_n}{\partial y} \right) \, dx_k
$$
$$
= \int_{\mathbb{R}} \left(\frac{y}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} - \frac{2 y p_0}{h^2} \frac{\partial h}{\partial x_k} \frac{\partial p_n}{\partial y} + \frac{y p_0}{h} \frac{\partial^2 p_n}{\partial y \partial x_k} \right) \, dx_k
$$
$$
= - \int_{\mathbb{R}} \left(\frac{y}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} + \frac{y p_0}{h} \frac{\partial^2 p_n}{\partial y \partial x_k} \right) \, dx_k.
$$

Combining these, we get

$$
\begin{aligned}
\mathbb{K}_{\mathbb{H}^{(k)}}(n) &= 4 \int_{\mathbb{R}^d} \int_0^\infty \left(\frac{y}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} + \frac{y p_0}{h^2} \frac{\partial h}{\partial y} \frac{\partial p_n}{\partial x_k} - \frac{y p_0}{h^2} \frac{\partial h}{\partial x_k} \frac{\partial p_n}{\partial y} \right) dy dx \\
&= 4 \int_{\mathbb{R}^d} \int_0^\infty \left(\frac{y}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} + \frac{p_0}{h} \frac{\partial p_n}{\partial x_k} + \frac{y}{h} \frac{\partial p_0}{\partial y} \frac{\partial p_n}{\partial x_k} + \frac{y p_0}{h} \frac{\partial^2 p_n}{\partial y \partial x_k} \right. \\
&\qquad\qquad \left. - \frac{y}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial p_n}{\partial y} - \frac{y p_0}{h} \frac{\partial^2 p_n}{\partial y \partial x_k} \right) dy dx \\
&= 4 \int_{\mathbb{R}^d} \int_0^\infty \left(\frac{p_0}{h} \frac{\partial p_n}{\partial x_k} + \frac{y}{h} \frac{\partial p_0}{\partial y} \frac{\partial p_n}{\partial x_k} \right) dy dx \\
&= 4 \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h} \frac{\partial p_n}{\partial x_k} \frac{\partial}{\partial y} \left(y p_0 \right) dy dx = -4 \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h} \frac{\partial p_0}{\partial x_k} \frac{\partial}{\partial y} \left(y p_n \right) dy dx
\end{aligned}
$$

We note that on the numerator of (5.9) we have $(|x-n|^2 - dy^2)$. Splitting this into $|x-n|^2 + y^2$ and $-(d+1)y^2$, then the first term can be absorbed in $S_n(x,y)$ and the other in $T_n(x,y)$. Thus, integration by parts with (5.8) and (5.9) gives that

(7.8)

$$
\begin{aligned}
\mathbb{K}_{\mathbb{H}^{(k)}}(n) &= 8 c_d^2 (d+1) \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h(x,y)} \frac{x_k y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}} (|x-n|^2 + y^2)^{\frac{d+1}{2}}} dy dx \\
&\quad - 4 c_d^2 (d+1)^2 \int_{\mathbb{R}^d} \int_0^\infty \frac{1}{h(x,y)} \frac{x_k y^4}{(|x|^2 + y^2)^{\frac{d+3}{2}} (|x-n|^2 + y^2)^{\frac{d+3}{2}}} dy dx.
\end{aligned}
$$

$\square$

Remark 7.3. *When $d = 1$, (4.7) gives $h(x,y) = \frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)}$, and*

$$
S_n(x,y) = \frac{4}{\pi^2} \frac{xy^2}{(x^2 + y^2)^2 ((x-n)^2 + y^2)},
$$

$$
T_n(x,y) = \frac{16}{3\pi^2} \frac{xy^4}{(x^2 + y^2)^2 ((x-n)^2 + y^2)^2},
$$

From this we have that for each $n \in \mathbb{Z} \setminus \{0\}$

$$
\begin{aligned}
\frac{\pi^2}{16} K_{\mathbb{H}}(n) &= \frac{\pi^2}{4} \iint \frac{S_n(x,y)}{h(x,y)} dx dy - \frac{3\pi^2}{16} \iint \frac{T_n(x,y)}{h(x,y)} dx dy \\
&= \int_0^\infty \coth(2\pi y) y^2 \left(\int_{\mathbb{R}} \frac{x}{(x^2 + y^2)^2 ((x-n)^2 + y^2)} dx \right) dy \\
&\quad - \int_0^\infty \frac{y^2}{\sinh(2\pi y)} \left(\int_{\mathbb{R}} \frac{x \cos(2\pi x)}{(x^2 + y^2)^2 ((x-n)^2 + y^2)} dx \right) dy \\
&\quad - \int_0^\infty \coth(2\pi y) y^4 \left(\int_{\mathbb{R}} \frac{x}{(x^2 + y^2)^2 ((x-n)^2 + y^2)^2} dx \right) dy \\
&\quad + \int_0^\infty \frac{y^4}{\sinh(2\pi y)} \left(\int_{\mathbb{R}} \frac{x \cos(2\pi x)}{(x^2 + y^2)^2 ((x-n)^2 + y^2)^2} dx \right) dy.
\end{aligned}
$$

The inner integrals with respect to x can all be computed using partial fraction decompositions. Indeed, for $n \in \mathbb{Z} \setminus \{0\}$ we have:

$$\int_{\mathbb{R}} \frac{x}{(x^2 + y^2)^2((x-n)^2 + y^2)} \, dx = \frac{2\pi n}{y(n^2 + 4y^2)^2},$$

$$\int_{\mathbb{R}} \frac{x \cos(2\pi x)}{(x^2 + y^2)^2((x-n)^2 + y^2)} \, dx = \frac{2\pi n e^{-2\pi y}}{y(n^2 + 4y^2)^2} - \frac{2\pi^2 e^{-2\pi y}}{n(n^2 + 4y^2)},$$

$$\int_{\mathbb{R}} \frac{x}{(x^2 + y^2)^2((x-n)^2 + y^2)^2} \, dx = \frac{\pi n(n^2 + 20y^2)}{2y^3(n^2 + 4y^2)^3},$$

$$\int_{\mathbb{R}} \frac{x \cos(2\pi x)}{(x^2 + y^2)^2((x-n)^2 + y^2)^2} \, dx = \frac{e^{-2\pi y} \pi n(n^2 + 20y^2)}{2y^3(n^2 + 4y^2)^3} + \frac{e^{-2\pi y} \pi^2(n^2 - 4y^2)}{ny^2(n^2 + 4y^2)^2}.$$

From these we obtain

$$K_{\mathbb{H}}(n) = \frac{1}{\pi n} \left(1 + 16\pi \int_0^\infty \frac{y^2 e^{-2\pi y}(3n^2 + 4y^2)}{\sinh(2\pi y)(n^2 + 4y^2)^2} \, dy \right)$$

$$= \frac{1}{\pi n} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2 y} \, dy \right),$$

where the last equality follows from integrations by parts with

$$f(y) = \frac{e^{-y}}{\sinh(y)}, \qquad g(y) = \frac{2y^3}{y^2 + \pi^2 n^2}.$$

This leads to the expression

$$(7.9) \qquad \mathbb{K}_{\mathbb{H}}(n) = \frac{1}{\pi n} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2(y)} \, dy \right) \mathbb{1}_{\mathbb{Z}\setminus\{0\}}(n)$$

$$= \frac{1}{\pi n} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2(y)} \, dy \right) \mathbb{1}_{\{|n| \geq 1\}}(n).$$

This formula was computed in [10] with a slightly different approach. We will return to this formula below in Section 9.

Recall that $R_{\text{dis}}^{(k)}$ are the CZ discrete Riesz transform given by

$$(7.10) \qquad R_{\text{dis}}^{(k)} f(m) = c_d \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \frac{n_k}{|n|^{d+1}} f(m - n) = \sum_{n \in \mathbb{Z}^d} K_{R_{\text{dis}}^{(k)}}(n) f(m - n),$$

where

$$K_{R_{\text{dis}}^{(k)}}(n) = c_d \frac{n_k}{|n|^{d+1}} \mathbb{1}_{\mathbb{Z}^d \setminus \{0\}}(n), \qquad c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}.$$

The following alternative representation for $K_{R_{\text{dis}}^{(k)}}$ will be frequently used below.

Proposition 7.4. *We have*

$$K_{R_{\text{dis}}^{(k)}}(n) = \left(\int_{\mathbb{R}^d} \int_0^\infty U_n(x,y)\,dydx \right) \mathbb{1}_{\{|n|\geq 1\}}(n) = \left(\int_{\mathbb{R}^d} \int_0^\infty S_n(x,y)\,dydx \right) \mathbb{1}_{\{|n|\geq 1\}}(n)$$

$$= \left(\int_{\mathbb{R}^d} \int_0^\infty T_n(x,y)\,dydx \right) \mathbb{1}_{\{|n|\geq 1\}}(n).$$

Proof. Let $N = \frac{d+3}{2}$. By the definition of Gamma function,

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{x_k y^2}{(|x|^2+y^2)^N(|x-n|^2+y^2)^{N-1}}\,dxdy$$

$$= \frac{1}{\Gamma(N)\Gamma(N-1)} \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty x_k y^2 u^{N-1} v^{N-2} e^{-u(|x|^2+y^2)-v(|x-n|^2+y^2)}\,dudvdxdy$$

$$= \frac{1}{\Gamma(N)\Gamma(N-1)} \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty x_k y^2 u^{N-1} v^{N-2} e^{-(u+v)|x-\frac{v}{u+v}n|^2-\frac{uv}{u+v}|n|^2-(u+v)y^2}\,dudvdxdy.$$

Since we have

$$\int_{\mathbb{R}^d} x_k e^{-(u+v)|x-\frac{v}{u+v}n|^2}\,dx = \int_{\mathbb{R}^d} \left(x_k + \frac{v}{u+v}n_k \right) e^{-(u+v)|x|^2}\,dx$$

$$= \frac{v}{u+v} n_k \int_{\mathbb{R}^d} e^{-(u+v)|x|^2}\,dx$$

$$= \frac{\pi^{\frac{d}{2}} v}{(u+v)^{1+\frac{d}{2}}} n_k$$

and

$$\int_0^\infty y^2 e^{-(u+v)y^2}\,dy = (u+v)^{-\frac{3}{2}} \frac{\sqrt{\pi}}{4},$$

it follows from Fubini's theorem that

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{x_k y^2}{(|x|^2+y^2)^N(|x-n|^2+y^2)^{N-1}}\,dxdy$$

$$= \frac{\pi^{\frac{d+1}{2}} n_k}{4\Gamma(N)\Gamma(N-1)} \int_0^\infty \int_0^\infty \frac{u^{N-1} v^{N-1}}{(u+v)^{N+1}} e^{-\frac{uv}{u+v}|n|^2}\,dudv.$$

Substituting $t = \frac{u}{u+v}$, $s = (u+v)|n|^2$, $dudv = |n|^{-4} sdsdt$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{x_k y^2}{(|x|^2+y^2)^N(|x-n|^2+y^2)^{N-1}}\,dxdy$$

$$= \frac{\pi^{\frac{d+1}{2}}}{4\Gamma(N)\Gamma(N-1)} \frac{n_k}{|n|^{d+1}} \int_0^1 \int_0^\infty s^{N-2} t^{N-1}(1-t)^{N-1} e^{-st(1-t)}\,dsdt$$

$$= \frac{1}{2(d+1)c_d} \frac{n_k}{|n|^{d+1}}.$$

Similarly, we have

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{x_k y^4}{(|x|^2 + y^2)^N (|x-n|^2 + y^2)^N} \, dxdy$$

$$= \frac{1}{\Gamma(N)^2} \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty x_k y^4 u^{N-1} v^{N-1} e^{-(u+v)|x - \frac{v}{u+v}n|^2 - \frac{uv}{u+v}|n|^2 - (u+v)y^2} \, dudvdxdy$$

$$= \frac{3\pi^{\frac{d+1}{2}}}{8\Gamma(N)^2} \frac{n_k}{|n|^{d+1}} \int_0^\infty \int_0^\infty \frac{u^{N-1} v^N}{(u+v)^{N+2}} e^{-\frac{uv}{u+v}} \, dudv$$

$$= \frac{3}{4c_d(d+1)^2} \frac{n_k}{|n|^{d+1}}.$$

$\square$

We are now ready to prove that the upper bound in Theorem 7.1 is sharp.

Theorem 7.5. *The ℓ^p-bound of $T_{\mathbb{H}^{(k)}}$ in Theorem 7.1 is best possible. That is, for all $d \geq 1$, $k = 1, \ldots d$, $\|T_{\mathbb{H}^{(k)}}\|_{\ell^p \to \ell^p} = \cot(\frac{\pi}{2p^*})$.*

The result will follow from the next two lemmas.

Lemma 7.6. *With the notation introduced earlier in this section, we have*

$$\lim_{|n| \to \infty} |n|^d |\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\text{dis}}^{(k)}}(n)| = 0.$$

Proof. Recall that

$$\mathbb{K}_{\mathbb{H}^{(k)}}(n) = \int_{\mathbb{R}^d} \int_0^\infty \frac{U_n(x,y)}{h(x,y)} \, dydx, \qquad K_{R_{\text{dis}}^{(k)}}(n) = \int_{\mathbb{R}^d} \int_0^\infty U_n(x,y) dydx,$$

where

$$U_n(x,y) = -4 \frac{\partial p_0}{\partial x_k}(x,y) \frac{\partial}{\partial y}(y p_n(x,y)).$$

Observe that $p_0(x,y) = \varepsilon^{d+1} p_0(\varepsilon x, \varepsilon y)$ and $p_n(x,y) = \varepsilon^{d+1} p_{\varepsilon n}(\varepsilon x, \varepsilon y)$. Therefore,

$$U_n(x,y) = \varepsilon^{2d+1} U_{\varepsilon n}(\varepsilon x, \varepsilon y);$$

here and below we abuse the notation and we allow n in p_n and U_n to be an arbitrary vector in $\mathbb{R}^d$. It follows that

$$\mathbb{K}_{\mathbb{H}^{(k)}}(n) = \int_{\mathbb{R}^d} \int_0^\infty \frac{U_n(x,y)}{h(x,y)} \, dydx$$

$$= \varepsilon^{2d+1} \int_{\mathbb{R}^d} \int_0^\infty \frac{U_{\varepsilon n}(\varepsilon x, \varepsilon y)}{h(x,y)} \, dydx$$

$$= \varepsilon^d \int_{\mathbb{R}^d} \int_0^\infty \frac{U_{\varepsilon n}(x,y)}{h(\frac{1}{\varepsilon}x, \frac{1}{\varepsilon}y)} \, dydx.$$

If we choose $\varepsilon = \frac{1}{|n|}$, we find that

$$|n|^d \mathbb{K}_{\mathbb{H}^{(k)}}(n) = \int_{\mathbb{R}^d} \int_0^\infty \frac{U_{n/|n|}(x,y)}{h(|n|x, |n|y)} \, dydx.$$

Similarly,

$$|n|^d K_{R_{\text{dis}}^{(k)}}(n) = \int_{\mathbb{R}^d} \int_0^\infty U_{n/|n|}(x,y) dy dx.$$

Therefore,

$$(7.11) \quad |n|^d |\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\text{dis}}^{(k)}}(n)| \leq \int_{\mathbb{R}^d} \int_0^\infty |U_{n/|n|}(x,y)| \times \left| \frac{1}{h(|n|x,|n|y)} - 1 \right| dy dx.$$

By the estimate (4.4), we have

$$\frac{y}{\sqrt{x^2+y^2}} \left(\frac{1}{h(|n|x,|n|y)} + 1 \right) \leq C_1$$

for a constant C_1 that depends only on the dimension d, and by Lemma 4.1, the left-hand side converges pointwise to zero as $|n| \to \infty$. On the other hand, by the explicit expression for $U_{n/|n|}$ given in Theorem 7.2, we have

$$\begin{aligned}
\frac{\sqrt{x^2+y^2}}{y} |U_{n/|n|}(x,y)| &\leq \frac{C_2\sqrt{x^2+y^2}}{y} \frac{|x_k|y^2}{(|x|^2+y^2)^{\frac{d+3}{2}}(|x-\frac{n}{|n|}|^2+y^2)^{\frac{d+1}{2}}} \\
&\quad + \frac{C_3\sqrt{x^2+y^2}}{y} \frac{|x_k|y^4}{(|x|^2+y^2)^{\frac{d+3}{2}}(|x-\frac{n}{|n|}|^2+y^2)^{\frac{d+3}{2}}} \\
&= C_2 \frac{|x_k|y}{(|x|^2+y^2)^{\frac{d+2}{2}}(|x-\frac{n}{|n|}|^2+y^2)^{\frac{d+1}{2}}} \\
&\quad + C_3 \frac{|x_k|y^3}{(|x|^2+y^2)^{\frac{d+2}{2}}(|x-\frac{n}{|n|}|^2+y^2)^{\frac{d+3}{2}}} \\
&\leq C_4 \frac{y}{(|x|^2+y^2)^{\frac{d+1}{2}}(|x-\frac{n}{|n|}|^2+y^2)^{\frac{d+1}{2}}}
\end{aligned}$$

for some constants C_2, C_3 and C_4 that again depend only on d. We have thus shown that

$$(7.12) \quad |U_{n/|n|}(x,y)| \times \left(\frac{1}{h(|n|x,|n|y)} + 1 \right) \leq C_1 C_4 \frac{y}{(|x|^2+y^2)^{\frac{d+1}{2}}(|x-\frac{n}{|n|}|^2+y^2)^{\frac{d+1}{2}}},$$

and additionally, as $|n| \to \infty$, the left-hand side converges point-wise to zero. Observe that if we denote by $\mathbb{O}_n$ an orthogonal transformation of $\mathbb{R}^d$ which maps $\frac{n}{|n|}$ to $e_1 = (1,0,0,\ldots,0)$, then the above estimate takes form

$$|U_{n/|n|}(\mathbb{O}_n x,y)| \times \left| \frac{1}{h(|n|\mathbb{O}_n x,|n|y)} - 1 \right| \leq C_1 C_4 \frac{y}{(|x|^2+y^2)^{\frac{d+1}{2}}(|x-e_1|^2+y^2)^{\frac{d+1}{2}}},$$

and the right-hand side no longer depends on n. Since the right-hand side is integrable, by the dominated convergence theorem we find that

$$\lim_{|n|\to\infty} \int_{\mathbb{R}^n} \int_0^\infty |U_{n/|n|}(x,y)| \times \left| \frac{1}{h(|n|x, |n|y)} - 1 \right| dxdy$$

$$= \lim_{|n|\to\infty} \int_{\mathbb{R}^n} \int_0^\infty |U_{n/|n|}(\mathbb{O}_n x, y)| \times \left| \frac{1}{h(|n|\mathbb{O}_n x, |n|y)} - 1 \right| dxdy = 0.$$

The desired result now follows from (7.11). □

Let us consider the continuous-discrete operator

$$\widetilde{T}_{\mathbb{H}^{(k)}} F(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} F(x+n) \mathbb{K}_{\mathbb{H}^{(k)}}(n).$$

Recall that $\mathbb{K}_{\mathbb{H}^{(k)}(0)} = 0$. By the argument of Theorem 6.2, the norm of the operator $\widetilde{T}_{\mathbb{H}^{(k)}}$ on $L^p(\mathbb{R}^d)$ is equal to the norm of the operator $T_{\mathbb{H}^{(k)}}$ on $\ell^p(\mathbb{Z}^d)$.

For $\varepsilon > 0$, $1 < p < \infty$ and a function F on $\mathbb{R}^d$, denote $\tau_\varepsilon F(x) = \varepsilon^{d/p} F(\varepsilon x)$ and $\widetilde{T}^\varepsilon_{\mathbb{H}^{(k)}} = \tau_{1/\varepsilon} \widetilde{T}_{\mathbb{H}^{(k)}} \tau_\varepsilon$. Observe that $\|\tau_\varepsilon F\|_p = \|F\|_p$, and hence the norm of the operator $\widetilde{T}^\varepsilon_{\mathbb{H}^{(k)}}$ on $L^p(\mathbb{R}^d)$ does not depend on ε.

We claim that as $\varepsilon \to 0^+$, the operators $\widetilde{T}^\varepsilon_{\mathbb{H}^{(k)}}$ approximate the continuous Riesz transform in (1.12). More precisely we have

Lemma 7.7. *Suppose that F is a smooth and compactly supported function on $\mathbb{R}^d$. Then*

$$\lim_{\varepsilon \to 0^+} \widetilde{T}^\varepsilon_{\mathbb{H}^{(k)}} F(x) = R^{(k)} F(x),$$

for every $x \in \mathbb{R}^d$.

Proof. We write

$$\widetilde{T}^\varepsilon_{\mathbb{H}^{(k)}} F(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} F(x+\varepsilon n) \mathbb{K}_{\mathbb{H}^{(k)}}(n)$$

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} (F(x+\varepsilon n) - F(x-\varepsilon n)) \mathbb{K}_{\mathbb{H}^{(k)}}(n)$$

(7.13)

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} (F(x+\varepsilon n) - F(x-\varepsilon n)) K_{R^{(k)}_{\mathrm{dis}}}(n)$$

$$+ \frac{1}{2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} (F(x+\varepsilon n) - F(x-\varepsilon n))(\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R^{(k)}_{\mathrm{dis}}}(n)).$$

We treat the two terms on the right-hand side separately.

Since $K_{R^{(k)}_{\mathrm{dis}}}(n) = \varepsilon^d K_{R^{(k)}}(\varepsilon n)$, the first term in the right-hand side of (7.13) is just the Riemann sum

$$\frac{\varepsilon^d}{2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} (F(x+\varepsilon n) - F(x-\varepsilon n)) K_{R^{(k)}}(\varepsilon n)$$

of the integral

$$R^{(k)}F(x) = \frac{1}{2}\int_{\mathbb{R}^d}(F(x+y) - F(x-y))K_{R^{(k)}}(y)dy.$$

By (6.5) of Theorem 6.2 applied to the kernels for $R^{(k)}$ we have,

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon^d}{2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}(F(x+\varepsilon n) - F(x-\varepsilon n))K_{R^{(k)}}(\varepsilon n)$$

$$= \frac{1}{2}\int_{\mathbb{R}^d}(F(x+y) - F(x-y))K_{R^{(k)}}(y)dy.$$

Thus, it remains to prove that the other term in the right-hand side of (7.13) converges to zero. To this end, we apply Lemma 7.6. First, there is a constant C_1 (which depends on F) such that $|F(x+y) - F(x-y)| \le C_1|y|$. If $R > 0$ is large enough, so that $F(x+y) = 0$ whenever $|y| \ge R$, we have

$$\frac{1}{2}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}(F(x+\varepsilon n) - F(x-\varepsilon n))(\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n))\right|$$

$$\le \frac{C_1}{2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\mathbb{1}_{B(0,R)}(\varepsilon n)|\varepsilon n|\,|\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n)|.$$

Given any $\delta > 0$, by Lemma 7.6 there is $r > 0$ such that $|\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n)| < \delta|n|^{-d}$ when $|n| \ge r$. Thus, denoting $C_2 = \sup_{n\ne 0}|\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n)|$, we find that

$$\frac{1}{2}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}(F(x+\varepsilon n) - F(x-\varepsilon n))(\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n))\right|$$

$$\le \frac{C_1\delta}{2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\mathbb{1}_{B(0,R)}(\varepsilon n)|\varepsilon n|\,|n|^{-d} + \frac{C_1C_2}{2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\mathbb{1}_{B(0,r)}(n)|\varepsilon n|$$

$$\le \frac{C_1\delta\varepsilon}{2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\mathbb{1}_{B(0,R/\varepsilon)}(n)|n|^{1-d} + \frac{C_1C_2\varepsilon}{2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\mathbb{1}_{B(0,r)}(n)|n|.$$

The first sum in the right-hand side is bounded by $\frac{C_3}{\varepsilon}$ for an appropriate constant C_3, and the second one does not depend on ε. Therefore,

$$\limsup_{\varepsilon\to 0^+}\frac{1}{2}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}(F(x+\varepsilon n) - F(x-\varepsilon n))(\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n))\right| \le \frac{C_1C_3\delta}{2}.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0^+}\frac{1}{2}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}(F(x+\varepsilon n) - F(x-\varepsilon n))(\mathbb{K}_{\mathbb{H}^{(k)}}(n) - K_{R_{\mathrm{dis}}^{(k)}}(n))\right| = 0,$$

and the proof is complete.   □

Applying Fatou's lemma, exactly as in the proof of Theorem 6.2, proves that $\|R^{(k)}\|_{L^p\to L^p} \le \|T_{\mathbb{H}^{(k)}}\|_{\ell^p\to\ell^p}$. This and the equality (1.15) give the assertion of Theorem 7.5.

Remark 7.8. *From the martingale inequality in* [16, Theorem 1] *we also obtain the following version of Essén's inequality for the probabilistic discrete Riesz transforms*

$$(7.14) \qquad \left\| \left(|T_{\mathbb{H}^{(k)}} f|^2 + |f|^2 \right)^{1/2} \right\|_{\ell^p} \le \sqrt{1 + \left(\cot\left(\frac{\pi}{2p^*} \right) \right)^2} \|f\|_{\ell^p}.$$

Let $F \in L^p(\mathbb{R}^d)$ be such that $F(x) = f(n)$ for $x \in n + Q$, $n \in \mathbb{Z}^d$. Then, it follows from Lemma 7.7 with Fatou's lemma that

$$\left\| \left(|\widetilde{T}_{\mathbb{H}^{(k)}} F|^2 + |F|^2 \right)^{1/2} \right\|_{L^p} \ge \left\| \left(|R^{(k)} F|^2 + |F|^2 \right)^{1/2} \right\|_{L^p}.$$

Let $\tau_\varepsilon F(x) = \varepsilon^{d/p} F(\varepsilon x)$. Since $\tau_{1/\varepsilon} R^{(k)} \tau_\varepsilon = R^{(k)}$ and $\|\tau_\varepsilon F\|_p = \|F\|_p$, we have

$$\left\| \left(|R^{(k)} F|^2 + |F|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left(|R^{(k)} (\tau_\varepsilon F)|^2 + |\tau_\varepsilon F|^2 \right)^{1/2} \right\|_{L^p}.$$

Since any function $G \in L^p(\mathbb{R}^d)$ can be approximated by $\tau_\varepsilon F$ where $F(x)$ is of the form $F(x) = \sum_{n \in \mathbb{Z}^d} f(n) \mathbb{1}_{n+Q}(x)$ for $f \in \ell^p(\mathbb{Z}^d)$, with the help of the Fatou's lemma, we see that the inequality (7.14) *is also sharp.*

Remark 7.9. *Similarly, using the matrices A_{jk} as in* [16, pg. 595] *would lead to what one may call "probabilistic discrete second order Riesz transforms" with ℓ^p-norms bounded above by $(p^* - 1)$. Notice, however, that even if we had the analogues of the above Lemmas for these operators (which we do not currently have), the $(p^* - 1)$ bound will not be sharp. Instead one would expect the sharp bound to be $\frac{1}{2}(p^* - 1)$ when $j \ne k$ and the Choi constant when $j = k$, see* [13, 36]. *Similar questions could be asked about the probabilistic discrete Beurling–Ahlfors operator, its sharp norm on ℓ^p and the relationships to the discrete Beurling–Ahlfors operator which Calderón and Zygmund highlight in their discussion on discrete singular integrals, see* [22, pg. 138]. *Based on Iwaniec's conjecture* [43] *that the norm of the Beurling–Ahlfors operator on $L^p(\mathbb{R}^2)$ is $(p^* - 1)$, $1 < p < \infty$, one would conjecture that the CZ discrete Beurling–Ahlfors operator should also have norm $(p^* - 1)$ on $\ell^p(\mathbb{Z}^2)$. We have not explored these questions.*

8. Fourier multiplier of the probabilistic discrete Hilbert transform

In this section, we focus on the case $d = 1$ and compute the Fourier transform of the probabilistic discrete Hilbert transform $T_{\mathbb{H}}$ whose kernel is given by

$$(8.1) \qquad \mathbb{K}_{\mathbb{H}}(n) = -4 \int_{\mathbb{R}} \int_0^\infty \frac{1}{h} \frac{\partial p}{\partial x} \frac{\partial}{\partial y} (y p_n) \, dy dx.$$

This representation for the kernel of the probabilistic discrete Hilbert transform $T_{\mathbb{H}}$ together with the computation from Proposition 7.4 makes it clear that there is a connection between this operator and the discrete Hilbert transform H_{dis}. However, this by itself does not yet give the bound $\|H_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \le \cot(\frac{\pi}{2p^*})$. In order to derive this bound from the bound of $\|T_{\mathbb{H}}\|_{\ell^p \to \ell^p}$, as already noted in the introduction, one needs to show that up to convolution with a probability kernel, the discrete Hilbert transform equals the probabilistic discrete Hilbert transform. This crucial fact was derived in [10, Lemma 1.3] using explicit computations to construct such a kernel. In what follows we provide a completely different proof of this fact, based on the formula from Lemmas 8.3, 8.4,

which gives an explicit formula for the Fourier transform of such a kernel. Although not clear at all at this point, it may be possible that such an approach based on the Fourier transform (as opposed to the complex variables approach in [10]) could lead to similar results for the discrete Riesz transforms $R_{\mathrm{dis}}^{(k)}$ in $d > 1$.

Again from (4.7) $h(x,y) = \frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)}$, which gives

$$(8.2) \qquad \frac{1}{h(x,y)} = \coth(2\pi y) - \frac{1}{\sinh(2\pi y)} \cos(2\pi x)$$

(a linear combination of 1 and $\cos(2\pi x)$ for y fixed). This is crucial for the computations below.

Let $\psi(x)$ be the digamma function defined by

$$\psi(x) = \frac{d}{dx} \log(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-y} - e^{-xy}}{1 - e^{-y}} \, dy$$

where γ is the Euler constant.

Theorem 8.1. *There exists a kernel $\mathcal{P}$ such that $\mathcal{P}(n) \geq 0$ for all $n \in \mathbb{Z}$, $\sum_{n \in \mathbb{Z}} \mathcal{P}(n) = 1$, and $H_{\mathrm{dis}} = T_{\mathbb{H}} * \mathcal{P}$. That is, for all $f : \mathbb{Z} \to \mathbb{R}$ of compact support,*

$$H_{\mathrm{dis}} f(n) = (T_{\mathbb{H}} * \mathcal{P}) f(n),$$

where $*$ denotes the convolution operation.

An immediate corollary of this is the main result in [10].

Corollary 8.2.

$$\|H_{\mathrm{dis}}\|_{\ell^p \to \ell^p} \leq \|T_{\mathbb{H}}\|_{\ell^p \to \ell^p} \leq \cot\left(\frac{\pi}{2p^*}\right).$$

Lemma 8.3. *The kernel for the probabilistic discrete Hilbert transform is given by*

$$\mathbb{K}_{\mathbb{H}}(n) = -i \int_{\mathbb{R}} \frac{\xi}{|\xi|} M(\xi) e^{2\pi i n \xi} \, d\xi$$

where

$$M(\xi) = \begin{cases} |\xi|^{-1}, & |\xi| \geq 1, \\ 1 + (1 - |\xi|) \left(2(\psi(1 + |\xi|) - \psi(1)) + |\xi|(\psi'(1 + |\xi|) - \psi'(1)), & |\xi| < 1. \end{cases}$$

Proof. We begin by observing that

$$\left(\frac{\partial p}{\partial x}(\cdot, y)\right)^{\wedge}(\xi) = 2\pi i \xi e^{-2\pi |\xi| y},$$

$$\left(\frac{\partial}{\partial y}(y p_n(\cdot, y))\right)^{\wedge}(\xi) = (1 - 2\pi |\xi| y) e^{-2\pi i n \cdot \xi} e^{-2\pi |\xi| y}.$$

Then,

$$\left(\frac{1}{h} \frac{\partial p}{\partial x}\right)^{\wedge}(\xi) = \coth(2\pi y) \left(\frac{\partial p}{\partial x}\right)^{\wedge}(\xi) - \frac{1}{\sinh(2\pi y)} \left(\cos(2\pi x) \frac{\partial p}{\partial x}\right)^{\wedge}(\xi)$$

and

$$\left(\cos(2\pi x)\frac{\partial p}{\partial x}\right)^{\wedge}(\xi) = \left(\frac{1}{2}(e^{2\pi ix} + e^{-2\pi ix})\frac{\partial p}{\partial x}\right)^{\wedge}(\xi)$$

$$= \pi i \left((\xi - 1)e^{-2\pi|\xi-1|y} + (\xi + 1)e^{-2\pi|\xi+1|y}\right)$$

$$= \begin{cases} 2\pi i\dfrac{\xi}{|\xi|}e^{-2\pi|\xi|y}(|\xi|\cosh(2\pi y) - \sinh(2\pi y)), & |\xi| \geq 1, \\[2ex] 2\pi i\dfrac{\xi}{|\xi|}e^{-2\pi y}(|\xi|\cosh(2\pi|\xi|y) - \sinh(2\pi|\xi|y)), & |\xi| < 1. \end{cases}$$

If $|\xi| \geq 1$, then

$$\left(\frac{1}{h}\frac{\partial p}{\partial x}\right)^{\wedge}(\xi) = 2\pi i\frac{\xi}{|\xi|}e^{-2\pi|\xi|y}.$$

On the other hand, if $|\xi| < 1$, then

$$\left(\frac{1}{h}\frac{\partial p}{\partial x}\right)^{\wedge}(\xi) = 2\pi i\frac{\xi}{|\xi|}e^{-2\pi|\xi|y}\left(|\xi| + (1 - |\xi|)\left(\frac{e^{4\pi|\xi|y} - 1}{e^{4\pi y} - 1}\right)\right).$$

Thus by Plancherel's theorem,

$$\mathbb{K}_{\mathbb{H}}(n) = -4\int_0^\infty \left(\int_{\mathbb{R}} \left(\frac{1}{h}\frac{\partial p}{\partial x}\right)^{\wedge}(\xi)\overline{\left(\frac{\partial}{\partial y}(yp_n(\cdot, y))\right)^{\wedge}}(\xi)\,d\xi\right) dy$$

$$= -8\pi i\int_{|\xi|\geq 1}\int_0^\infty \frac{\xi}{|\xi|}e^{-4\pi|\xi|y}(1 - 2\pi|\xi|y)e^{2\pi in\cdot\xi}\,dyd\xi$$

$$- 8\pi i\int_{|\xi|\leq 1}\int_0^\infty \frac{\xi}{|\xi|}e^{-4\pi|\xi|y}(1 - 2\pi|\xi|y)e^{2\pi in\cdot\xi}\left(|\xi| + (1 - |\xi|)\left(\frac{e^{4\pi|\xi|y} - 1}{e^{4\pi y} - 1}\right)\right)\,dyd\xi$$

$$= -i\int_{|\xi|\geq 1}\frac{\xi}{|\xi|^2}e^{2\pi in\cdot\xi}\,d\xi - i\int_{|\xi|\leq 1}\frac{\xi}{|\xi|}e^{2\pi in\cdot\xi}\,d\xi$$

$$- 2i\int_{|\xi|\leq 1}\frac{\xi}{|\xi|}e^{2\pi in\cdot\xi}\left((1 - |\xi|)\int_0^\infty\left(1 - \frac{|\xi|y}{2}\right)\frac{e^{-y} - e^{-(1+|\xi|)y}}{1 - e^{-y}}\,dy\right)d\xi.$$

Since

$$\int_0^\infty (1 - \frac{|\xi|y}{2})\frac{e^{-y} - e^{-(1+|\xi|)y}}{1 - e^{-y}}\,dy = (\psi(1 + |\xi|) - \psi(1)) + \frac{|\xi|}{2}(\psi'(1 + |\xi|) - \psi'(1)),$$

where ψ is the digamma function, we obtain that

$$\mathbb{K}_{\mathbb{H}}(n) = -i\int_{\mathbb{R}}\frac{\xi}{|\xi|}M(\xi)e^{2\pi in\xi}\,d\xi.$$

□

Lemma 8.4. *For $\xi \in Q = [-\frac{1}{2}, \frac{1}{2})$ and a compactly supported function f on $\mathbb{Z}$, the Fourier transform of the probabilistic discrete Hilbert transform $T_{\mathbb{H}}(f)$ is given by*

$$\mathcal{F}(T_{\mathbb{H}}(f))(\xi) = -i\frac{\xi}{|\xi|}\widetilde{M}(\xi)\mathcal{F}(f)(\xi)$$

where

$$\widetilde{M}(\xi) = 1 + (1 - 2|\xi|)(\psi(1 + |\xi|) + \psi(1 - |\xi|) - 2\psi(1)) + |\xi|(1 - |\xi|)(\psi'(1 + |\xi|) - \psi'(1 - |\xi|)).$$

Proof. By Lemma 8.3, we have

$$\mathbb{K}_{\mathbb{H}}(n) = -i\int_{\mathbb{R}} \frac{\xi}{|\xi|}M(\xi)e^{2\pi in\xi}\,d\xi = -i\sum_{m\in\mathbb{Z}}\int_{Q} \frac{\xi + m}{|\xi + m|}M(\xi + m)e^{2\pi in\xi}\,d\xi.$$

Thus, it suffices to show that

$$\sum_{n\in\mathbb{Z}} \frac{\xi + n}{|\xi + n|}M(\xi + n) = \frac{\xi}{|\xi|}\widetilde{M}(\xi),$$

for $\xi \in [0, \frac{1}{2}]$. By the series representation for the digamma function

$$\psi(1 + z) = -\gamma + \sum_{m=1}^{\infty}\left(\frac{1}{m} - \frac{1}{m + z}\right),$$

we have

$$\sum_{|n|\geq 2} \frac{\xi + n}{|\xi + n|}M(\xi + n) = \sum_{m=2}^{\infty}\left(\frac{1}{\xi + m} + \frac{1}{\xi - m}\right)$$

$$= \sum_{m=1}^{\infty}\left(\frac{1}{m + \xi} - \frac{1}{m} + \frac{1}{m} - \frac{1}{m - \xi}\right) + \frac{2\xi}{1 - \xi^2}$$

$$= \psi(1 - \xi) - \psi(1 + \xi) + \frac{2\xi}{1 - \xi^2}.$$

Using the recurrence relation $\psi(1 + z) = \psi(z) + \frac{1}{z}$, we have

$$\frac{\xi + 1}{|\xi + 1|}M(\xi + 1) + \frac{\xi - 1}{|\xi - 1|}M(\xi - 1)$$

$$= \frac{1}{\xi + 1} - \left(1 + 2\xi(\psi(2 - \xi) - \psi(1)) + \xi(1 - \xi)(\psi'(2 - \xi) - \psi'(1))\right)$$

$$= 1 - \frac{2\xi}{1 - \xi^2} - \left(1 + 2\xi(\psi(1 - \xi) - \psi(1)) + \xi(1 - \xi)(\psi'(1 - \xi) - \psi'(1))\right).$$

Therefore, for $\xi \in [0, \frac{1}{2}]$, we have

$$\sum_{n \in \mathbb{Z}} \frac{\xi + n}{|\xi + n|} M(\xi + n) = M(\xi) + \sum_{|n| \geq 2} \frac{\xi + n}{|\xi + n|} M(\xi + n) + \frac{\xi + 1}{|\xi + 1|} M(\xi + 1) + \frac{\xi - 1}{|\xi - 1|} M(\xi - 1)$$

$$= 1 + (1 - \xi)\left(2(\psi(1 + \xi) - \psi(1)) + \xi(\psi'(1 + \xi) - \psi'(1))\right.$$

$$+ \psi(1 - \xi) - \psi(1 + \xi) + \frac{2\xi}{1 - \xi^2} + 1 - \frac{2\xi}{1 - \xi^2}$$

$$- \left(1 + 2\xi(\psi(1 - \xi) - \psi(1)) + \xi(1 - \xi)(\psi'(1 - \xi) - \psi'(1))\right)$$

$$= \widetilde{M}(\xi),$$

which completes the proof.

$\square$

Lemma 8.5. *Let $g(\xi) = \frac{1 - 2\xi}{\widetilde{M}(\xi)}$ on $\xi \in [0, \frac{1}{2}]$. Then, g is strictly positive, decreasing, and convex on $[0, \frac{1}{2}]$ with $g(0) = 1$, $g'(0) = -2$, $g'(\frac{1}{2}) = 0$.*

Proof. First, we claim that $\widetilde{M}(\xi) \geq 0$. Since $\widetilde{M}(0) = 1$, and $\widetilde{M}(1/2) = 0$, it suffices to show that $\widetilde{M}(\xi)$ is decreasing in $[0, \frac{1}{2}]$. Let $\varphi(x) := \psi(1 + x) + \psi(1 - x) - 2\psi(1)$ where ψ is the digamma function. Then, $\widetilde{M}$ can be written as

$$\widetilde{M}(\xi) = 1 + (1 - 2|\xi|)\varphi(|\xi|) + |\xi|(1 - |\xi|)\varphi'(|\xi|).$$

By the definition of ψ, $\varphi(x)$ has the integral representation

$$\varphi(x) = 2 \int_0^\infty \frac{1 - \cosh(xy)}{e^y - 1} \, dy.$$

It follows from this that $\frac{d^n}{dx^n}\varphi(x) \leq 0$ for all $n = 0, 1, 2, \cdots$ and $x \in [0, \frac{1}{2}]$. Then,

$$\varphi(x) = -2x^2 \sum_{n=1}^\infty \frac{1}{n(n^2 - x^2)}, \qquad \varphi'(x) = -4x \sum_{n=1}^\infty \frac{n}{(n^2 - x^2)^2}$$

$$\varphi''(x) = -4 \sum_{n=1}^\infty \frac{n(n^2 + 3x^2)}{(n^2 - x^2)^3}.$$

Thus,

$$\widetilde{M}'(x) = -2\varphi(x) + 2(1 - 2x)\varphi'(x) + x(1 - x)\varphi''(x)$$

$$= -4 \sum_{n=1}^\infty \frac{1}{n(n^2 - x^2)^3} \left(-x^2(n^2 - x^2)^2 + 2(1 - 2x)n^2(n^2 - x^2) + x(1 - x)n^2(n^2 + 3x^2)\right).$$

The numerator in the summand can be written as

$$- x^2(n^2 - x^2)^2 + 2(1 - 2x)n^2(n^2 - x^2) + x(1 - x)n^2(n^2 + 3x^2)$$

$$= (1 - 2x)(2 + x)n^4 - x^2(x^2 - 7x + 2)n^2 - x^6$$

$$= (1 - 2x)(2 + x)(n^4 - n^2) + ((1 - 2x)(2 + x) - x^2(x^2 - 7x + 2) - x^6)n^2 + x^6(n^2 - 1)$$

$$= (1 - 2x)(2 + x)(n^4 - n^2) - (x - 1)^2(x^4 + 2x^3 + 4x^2 - x - 2)n^2 + x^6(n^2 - 1).$$

Since $x^4 + 2x^3 + 4x^2 - x - 2 \le 0$ for all $x \in [0, \frac{1}{2}]$, we conclude that $\widetilde{M}'(x) \le 0$ for $x \in [0, \frac{1}{2}]$ and so $\widetilde{M}(x) \ge 0$ on $[0, \frac{1}{2}]$. Since

$$g(1/2) = \lim_{x \to \frac{1}{2}^-} \frac{1 - 2x}{\widetilde{M}(x)} = -\lim_{x \to \frac{1}{2}^-} \frac{2}{\widetilde{M}'(x)} = \frac{1}{\frac{7}{2}\zeta(3) - 4\log 2} > 0,$$

g is strictly positive.

Let $u(x) := \widetilde{M}(x)/(1 - 2|x|)$. We claim that u is increasing and concave on $[0, 1/2]$. Denote $\varphi(x) = \psi(1 + x) + \psi(1 - x) - 2\psi(1)$. Since

$$u(x) = \frac{1}{1 - 2x} + \varphi(x) + \frac{x(1 - x)}{1 - 2x}\varphi'(x)$$

$$= \frac{1}{4(1 - 2x)}(4 + \varphi'(x)) + \varphi(x) - \frac{1 - 2x}{4}\varphi'(x),$$

$$u'(x) = \frac{1}{2(1 - 2x)^2}(4 + \varphi'(x)) + \frac{\varphi''(x)}{4(1 - 2x)} + \frac{3}{2}\varphi'(x) - \frac{1 - 2x}{4}\varphi''(x),$$

it follows from L'hospital's rule and the recurrence property of polygamma functions that

$$u'(1/2) = -\frac{1}{16}\varphi'''(1/2) + \frac{3}{2}\varphi'(1/2) = 0.$$

Thus, it suffices to prove that $u''(x) \le 0$ for $x \in [0, \frac{1}{2}]$. Let

$$v(x) := (1 - 2x)^3 u''(x)$$

$$= 2(4 + \varphi'(x)) + (1 - 2x)(1 + 2(1 - 2x)^2)\varphi''(x) + x(1 - x)(1 - 2x)^2\varphi'''(x).$$

Note that

$$v'(x) = (1 - 2x)^2(-12\varphi^{(2)}(x) + 4(1 - 2x)\varphi^{(3)}(x) + x(1 - x)\varphi^{(4)}(x)).$$

Using the series representation for $\varphi(x)$ (see the remark above), we get

$$-12\varphi^{(2)}(x) + 4(1 - 2x)\varphi^{(3)}(x) + x(1 - x)\varphi^{(4)}(x)$$

$$= \sum_{n=1}^{\infty} \frac{n}{(n^2 - x^2)^5}(n^6 - 5x(1 - 2x)n^4 - 5x^3(2 - x)n^2 - x^5)$$

$$= \sum_{n=1}^{\infty} \frac{n}{(n^2 - x^2)^5}((n^6 - n^4) + (1 - x)^5 n^4 + 5x^3(2 - x)(n^4 - n^2) + (n^4 - 1)x^5) \ge 0$$

for all $x \in [0, \frac{1}{2}]$. Since $v'(x) \ge 0$ and $v(1/2) = 0$, we obtain that $v(x) \le 0$ and so $u''(x) \le 0$ for $x \in [0, \frac{1}{2}]$ as desired.

$\square$

Remark 8.6. *From Lemma 8.5, we see that* $\|\widetilde{M}\|_\infty = 1$. *This implies that* $T_{\mathbb{H}}$ *is bounded in* $\ell^2(\mathbb{Z})$ *and its norm is 1, as we already know.*

Proof of Theorem 8.1. The proof is based on Pólya's criterion for discrete random variables (see for example [19, p.353]). The Fourier multiplier for the classical discrete

Hilbert transform H_{dis} with kernel $K_{H_{\mathrm{dis}}}(n) = \frac{1}{\pi n}$ is

$$\mathcal{F}(K_{H_{\mathrm{dis}}})(\xi) = \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{\pi n} e^{-2\pi i n\xi} = -\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{2i}{\pi n}\sin(2\pi n\xi) = -i\frac{\xi}{|\xi|}(1-2|\xi|)$$

for $\xi \in [-\frac{1}{2}, -\frac{1}{2})$. Thus, we have

$$\mathcal{F}(K_{H_{\mathrm{dis}}})(\xi) = -i\frac{\xi}{|\xi|}(1-2|\xi|) = \mathcal{F}(\mathbb{K}_{\mathbb{H}})(\xi)g(\xi), \qquad g(\xi) = \frac{1-2|\xi|}{\widetilde{M}(\xi)}.$$

Since we know $g(\xi)$ is strictly positive, convex, decreasing on $[0, \frac{1}{2}]$ and symmetric on Q by Lemma 8.5, there exists a sequence of continuous, symmetric, piecewise linear functions $g_j(\xi)$ on Q converging to $g(\xi)$ pointwise. In particular, for each j, there exist $0 = t_1 < t_2 < \cdots < t_k = \frac{1}{2}$ such that $g_j(0) = 1$, $g_j(\frac{1}{2}) = g(\frac{1}{2}) =: c$, and $g_j(\xi) = a_i - s_i\xi$ for $\xi \in [t_{i-1}, t_i]$, for some $a_i, s_i \geq 0$, $i = 1, 2, \cdots, k$. Let $\eta(\xi) = \max\{0, 1 - |\xi|\}$, $\eta_t(\xi) := \eta(\xi/t)$, and $\eta_\infty(\xi) = 1$ on Q. Since g_j is continuous and $g(0) = 1, g(\frac{1}{2}) = c$,

$$1 = \sum_{i=1}^{k} s_i(t_i - t_{i-1}) + c = \sum_{i=1}^{k} t_i(s_i - s_{i+1}) + c$$

by letting $s_{k+1} = 0$. One can see that

$$g_j(\xi) = \sum_{i=1}^{k} t_i(s_i - s_{i+1})\eta_{t_i}(\xi) + c\eta_\infty(\xi).$$

Since η_t, η_∞ for $0 < t \leq \frac{1}{2}$ are the characteristic functions of discrete random variables and g_j is the convex combinations of them, g_j is also the characteristic function of a discrete random variable X_j on $\mathbb{Z}$. Since g_j converges to g pointwise and g is continuous at 0, we see that g is also the characteristic function of a random variable X. Note that

$$i\frac{\xi}{|\xi|} = \mathcal{F}(\mathcal{U})(\xi), \qquad \mathcal{U}(n) = \begin{cases} -\frac{1}{2\pi n}, & n \text{ is odd,} \\ 0, & n \text{ is even.} \end{cases}$$

We know that $\mathcal{F}(\mathbb{K}_{\mathbb{H}})(\xi) = -i\frac{\xi}{|\xi|}\widetilde{M}(\xi)$, and $\mathbb{K}_{\mathbb{H}}(n)$ is decreasing in n with $|\mathbb{K}_{\mathbb{H}}(n)| \leq \frac{C}{n}$ for some $C > 0$. Thus, the convolution $\mathbb{K}_{\mathbb{H}} * \mathcal{U}(n)$ is well-defined and the Fourier transform is $\widetilde{M}$. Since $\widetilde{M}(0) = 1$, we obtain that $\widetilde{M}(\xi)$ is the characteristic function of a discrete random variable, say Y, on $\mathbb{Z}$. Assume X and Y are independent, then the characteristic function of the sum $X + Y$ is $1 - 2|\xi|$, which is the characteristic function of a discrete random variable. So, we conclude that X is also discrete, supported on $\mathbb{Z}$. That is, there exists a probability kernel $\mathcal{P}$ such that $\mathcal{P}(n) \geq 0$ for all $n \in \mathbb{Z}$, $\sum_{n\in\mathbb{Z}} \mathcal{P}(n) = 1$, and $g(\xi) = \mathcal{F}(\mathcal{P})(\xi)$ on Q. $\square$

Question 8.7. *Does Theorem 8.1 hold for $d > 1$? More precisely, is there a probability kernel $\mathcal{P}^{(k)}$ on $\mathbb{Z}^d$ such that*

$$R_{\mathrm{dis}}^{(k)}f(n) = \sum_{m\in\mathbb{Z}^d} \mathcal{P}^{(k)}(n-m)T_{\mathbb{H}^{(k)}}f(m)?$$
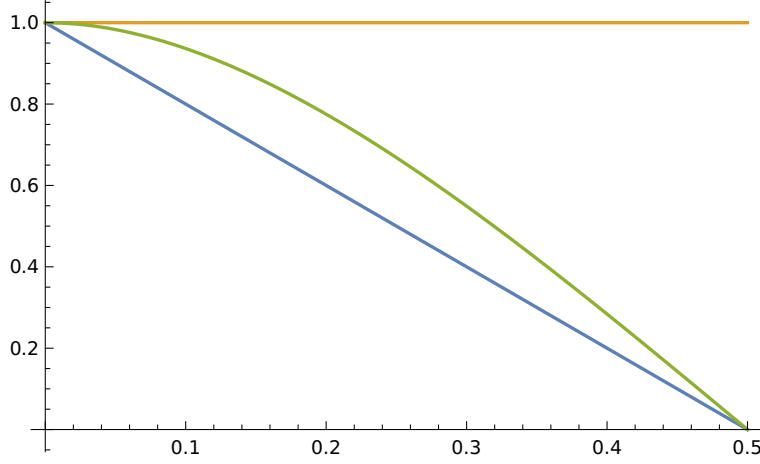
FIGURE 1. Comparison between the Fourier multipliers for the classical Hilbert transform (orange), the discrete Hilbert transform (blue), and the probabilistic discrete Hilbert transform (green).

9. PROBABILISTIC CONTINUOUS RIESZ TRANSFORMS

Given that discrete operators obtained from Calderón–Zygmund kernels as defined in (6.3) are simply defined by replacing the continuous variable z by the discrete variable n and avoiding the singularity at $\{0\}$ in the sum are bounded on ℓ^p (Propostion 6.1), it is natural to ask if the opposite is also true in the current situation. More precisely, is it true that the kernels obtained from $\mathbb{K}_{\mathbb{H}^{(k)}}$ simply by replacing $n \in \mathbb{Z}^d$ with $z \in \mathbb{R}^d$, $|z| \geq 1$, together with the some modification for $|z| < 1$, are Calderón–Zygmund kernels satisfying (6.3)? In this section we give a formula for such continuous kernels that satisfy (6.3), with the exception of the C^1 property on the sphere $|z| = 1$, and are also bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$ with rather precise norm bounds. Since we are able to find various explicit constants for the case $d = 1$, we consider the cases $d = 1$ and $d > 1$ separately.

From formula (1.8) a natural version of a continuous kernel which gives the probabilistic discrete Hilbert transform à la Calderón–Zygmund would be

(9.1)

$$\mathbb{K}_{\mathbb{H}}(z) = \frac{1}{\pi z}\left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 z^2)\sinh^2(y)}\,dy\right)\mathbb{1}_{\{|z|\geq 1\}}(z) + \frac{1}{\pi z}\mathbb{1}_{\{|z|<1\}}(z), \quad z \in \mathbb{R}.$$

Similarly, for $d > 1$ from (7.8) a natural definition of the version of a continuous kernel which gives the probabilistic discrete Riesz transforms for $k = 1, 2, \ldots, d$ would be

(9.2)

$$\mathbb{K}_{\mathbb{H}^{(k)}}(z) := \left(-4\int_{\mathbb{R}^d}\int_0^\infty \frac{1}{h(x,y)}\frac{\partial p_0}{\partial x_k}\frac{\partial}{\partial y}(yp_z)\,dydx\right)\mathbb{1}_{\{|z|\geq 1\}}(z) + c_d\frac{z_k}{|z|^{d+1}}\mathbb{1}_{\{|z|<1\}}(z)$$
$$= \left(I_1^{(k)}(z) + I_2^{(k)}(z)\right)\mathbb{1}_{\{|z|\geq 1\}}(z) + c_d\frac{z_k}{|z|^{d+1}}\mathbb{1}_{\{|z|<1\}}(z),$$

where

$$I_1^{(k)}(z) = \int_{\mathbb{R}^d} \int_0^\infty \frac{8c_d^2(d+1)x_k y^2}{h(x,y)(|x|^2+y^2)^{\frac{d+3}{2}}(|x-z|^2+y^2)^{\frac{d+1}{2}}}\, dxdy,$$

and

$$I_2^{(k)}(z) = -\int_{\mathbb{R}^d} \int_0^\infty \frac{4c_d^2(d+1)^2 x_k y^4}{h(x,y)(|x|^2+y^2)^{\frac{d+3}{2}}(|x-z|^2+y^2)^{\frac{d+3}{2}}}\, dxdy$$

and $c_d = \Gamma(\frac{d+1}{2})\pi^{-\frac{d+1}{2}}$.

Notice that $\mathbb{K}_{\mathbb{H}^{(k)}}$ is not continuous on $|z|=1$ and hence not a Calderón–Zygmund kernel requiring (6.3). Nevertheless, with these definitions we have

Theorem 9.1. *For any $d \geq 1$ and $k = 1, \ldots d$, the kernels $\mathbb{K}_{\mathbb{H}^{(k)}}(z)$ ($\mathbb{K}_{\mathbb{H}}(z)$ when $d=1$) satisfy*

  (i)

(9.3)
$$|\mathbb{K}_{\mathbb{H}^{(k)}}(z)| \leq \frac{C_d}{|z|^d}, \quad z \in \mathbb{R}^d \setminus \{0\}.$$

  (ii)

(9.4)
$$|\nabla \mathbb{K}_{\mathbb{H}^{(k)}}(z)| \leq \frac{C_d}{|z|^{d+1}}, \quad z \in \mathbb{R}^d \setminus \{0\}, \quad |z| \neq 1,$$

  where C_d depends only on d. Furthermore,
  (iii) For $d = 1$,

(9.5)
$$\sup_{\xi \in \mathbb{R}} |\widehat{\mathbb{K}_{\mathbb{H}}}(\xi)| \leq 1 + \frac{2}{\pi}\int_0^\infty \frac{y\ln(y^2/\pi^2+1)}{\sinh^2(y)}\, dy \approx 1.09956.$$

  (iv) For $d > 1$ and all $k = 1, \ldots d$, we have

(9.6)
$$|\widehat{\mathbb{K}_{\mathbb{H}^{(k)}}}(\xi)| \leq C_d, \quad \text{for all} \quad \xi \in \mathbb{R}^d \setminus \{0\}$$

Proof. We first show the case $d = 1$ which is computationally much simpler and will give explicit constants, particularly the bound for the Fourier transform. Clearly $|\mathbb{K}_{\mathbb{H}}(z)| = \frac{1}{\pi|z|}$, for $0 < |z| < 1$. On the other hand,

$$\int_0^\infty \frac{y^3}{(y^2+\pi^2 z^2)\sinh^2(y)}\, dy \leq \frac{1}{\pi^2 z^2}\int_0^\infty \frac{y^3}{\sinh^2(y)}\, dy$$

$$= \frac{4}{\pi^2 z^2}\int_0^\infty \frac{y^3}{e^{2y}(1-e^{-2y})^2}\, dy$$

$$= \frac{4}{\pi^2 z^2}\sum_{k=1}^\infty k\int_0^\infty y^3 e^{-2ky}\, dy$$

(9.7)
$$= \frac{3}{2\pi^2 z^2}\sum_{k=1}^\infty \frac{1}{k^3} = \frac{3}{2\pi^2 z^2}\zeta(3).$$

From this it follows that $|\mathbb{K}_{\mathbb{H}}(z)| \leq \frac{C_1}{|z|}$, for all $|z| > 0$. Similarly, for $|z| \geq 1$,

$$|\mathbb{K}'_{\mathbb{H}}(z)| = \left| -\frac{1}{\pi z^2} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 z^2)\sinh^2(y)} \, dy \right) - \int_0^\infty \frac{4y^3}{(y^2 + \pi^2 z^2)^2 \sinh^2(y)} \, dy \right|$$

$$\leq \frac{C_2}{|z|^2},$$

and again we have $|\mathbb{K}'_{\mathbb{H}}(z)| \leq \frac{C}{|z|^2}$ for all $|z| > 0$, $|z| \neq 1$. Here, C is a universal constant. Thus $\mathbb{K}_{\mathbb{H}}$ satisfies (i) and (ii).

In addition, we have, in the principal value sense,

$$\widehat{\mathbb{K}}_{\mathbb{H}}(\xi) = \int_{\mathbb{R}} \mathbb{K}_{\mathbb{H}}(z) e^{-2\pi i z \cdot \xi} dz$$

(9.8)
$$= \int_{\mathbb{R}} \frac{1}{\pi z} e^{-2\pi i z \cdot \xi} dz + \frac{1}{\pi} \int_0^\infty \frac{2y^3}{\sinh^2(y)} \left(\int_{\{|z|>1\}} \frac{e^{-2\pi i z \cdot \xi}}{z(y^2 + \pi^2 z^2)} \, dz \right) dy$$

$$= -i \operatorname{sign}(\xi) + \frac{1}{\pi} \int_0^\infty \frac{2y^3}{\sinh^2(y)} \left(\int_{\{|z|>1\}} \frac{e^{-2\pi i z \cdot \xi}}{z(y^2 + \pi^2 z^2)} \, dz \right) dy$$

$$= -i \operatorname{sign}(\xi) \left(1 + \frac{1}{\pi} \int_0^\infty \frac{2y^3}{\sinh^2(y)} \left(\int_{\{|z|>1\}} \frac{\sin(2\pi z|\xi|)}{z(y^2 + \pi^2 z^2)} \, dz \right) dy \right).$$

By Fubini and the following identity obtained by integration by parts,

$$\int_1^\infty \frac{1}{z(y^2 + \pi^2 z^2)} dz = \frac{\ln(y^2/\pi^2 + 1)}{2y^2},$$

we have

$$\left| \frac{1}{\pi} \int_0^\infty \frac{y^3}{\sinh^2(y)} \left(\int_{\{|z|>1\}} \frac{e^{-2\pi i z \cdot \xi}}{z(y^2 + \pi^2 z^2)} \, dz \right) dy \right| \leq \frac{2}{\pi} \int_0^\infty \frac{y^3}{\sinh^2(y)} \left(\int_1^\infty \frac{1}{z(y^2 + \pi^2 z^2)} \, dz \right) dy$$

$$= \frac{1}{\pi} \int_0^\infty \frac{y \ln(y^2/\pi^2 + 1)}{\sinh^2(y)} dy$$

(9.9)
$$\approx 0.0497822.$$

Hence for all $\xi \in \mathbb{R}$,

$$|\widehat{\mathbb{K}}_{\mathbb{H}}(\xi)| \leq 1 + \frac{2}{\pi} \int_0^\infty \frac{y \ln(y^2/\pi^2 + 1)}{\sinh^2(y)} dy \approx 1.09956,$$

which is the claim in (iii).

We now suppose $d > 1$. By (7.12), (4.4) and a change of variables we have, for $|z| \geq 1$ and $z = |z|\theta$, that

$$|\mathbb{K}_{\mathbb{H}^{(k)}}(z)| \leq C_d \int_0^\infty \int_{\mathbb{R}^d} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy$$

$$= \frac{C_d}{|z|^d} \int_0^\infty \int_{\mathbb{R}^d} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (|x - \theta|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy.$$

Since $\{|x - \theta| \leq \frac{1}{2}\} \subset \{|x| > \frac{1}{2}\}$, we have

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (|x - \theta|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy$$

$$= \int_0^\infty \int_{\{|x-\theta| \leq \frac{1}{2}\}} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (|x - \theta|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy$$

$$+ \int_0^\infty \int_{\{|x-\theta| > \frac{1}{2}\}} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (|x - \theta|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy$$

$$\leq \int_0^\infty \int_{\{|x| > \frac{1}{2}\}} \frac{y}{(\frac{1}{4} + y^2)^{\frac{d+1}{2}} (|x - \theta|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy$$

$$+ \int_0^\infty \int_{\{|x-\theta| > \frac{1}{2}\}} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (\frac{1}{4} + y^2)^{\frac{d+1}{2}}} \, dx dy.$$

By change of variables,

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}} (|x - \theta|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy$$

$$\leq 2 \int_0^\infty \left(\int_{\mathbb{R}^d} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}}} \, dx \right) \frac{1}{(\frac{1}{4} + y^2)^{\frac{d+1}{2}}} \, dy$$

$$\leq 2 \int_0^\infty \left(\int_{\mathbb{R}^d} \frac{1}{(|x|^2 + 1)^{\frac{d+1}{2}}} \, dx \right) \frac{1}{(\frac{1}{4} + y^2)^{\frac{d+1}{2}}} \, dy < \infty.$$

This together with the obvious bound for the second term in (9.2) gives that $|\mathbb{K}_{\mathbb{H}^{(k)}}(z)| \leq \frac{C_d}{|z|^d}$, for $|z| > 0$. Next, for $j = 1, 2, \cdots, d$, differentiation and (4.4) gives that for $|z| > 1$,

$$\left| \frac{\partial}{\partial z_j} I_1^{(k)}(z) \right| = C_d \left| \int_0^\infty \int_{\mathbb{R}^d} \frac{x_k y^2 (x_j - z_j)}{h(x,y)(|x|^2 + y^2)^{\frac{d+3}{2}} (|x - z|^2 + y^2)^{\frac{d+3}{2}}} \, dx dy \right|$$

$$\leq \frac{C_d}{|z|^{(d+1)}} \int_0^\infty \int_{\mathbb{R}^d} \frac{|x_k x_j| y}{(|x|^2 + y^2)^{\frac{d+2}{2}} (|x - \theta|^2 + y^2)^{\frac{d+3}{2}}} \, dx dy$$

$$+ \frac{C_d |z_j|}{|z|^{(d+2)}} \int_0^\infty \int_{\mathbb{R}^d} \frac{|x_k z_j| y}{(|x|^2 + y^2)^{\frac{d+2}{2}} (|x - \theta|^2 + y^2)^{\frac{d+3}{2}}} \, dx dy$$

$$\leq \frac{C_d}{|z|^{d+1}},$$

where $z = |z| \theta$. Similarly, we can obtain the same upper bound for $\frac{\partial}{\partial z_j} I_2^{(k)}(z)$, $|z| > 1$, which leads to $\left| \nabla \mathbb{K}_{\mathbb{H}^{(k)}}(z) \right| \leq \frac{C_d}{|z|^{d+1}}$ for all $|z| > 0$ and $|z| \neq 1$.

It remains to show that the Fourier transform of $\mathbb{K}_{\mathbb{H}^{(k)}}(z)$ is bounded. By (7.8) and Proposition 7.4, we have that

$$(9.10) \qquad -4 \int_{\mathbb{R}^d} \int_0^\infty \frac{\partial p_0(x,y)}{\partial x_k} \frac{\partial}{\partial y}(y p_z(x,y)) \, dy dx = \frac{c_d z_k}{|z|^{d+1}}, \quad |z| > 0.$$

This formula can also be easily verified using the Fourier transform. More precisely, for a smooth function ϕ with compact support, we have

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \int_0^\infty \frac{\partial p}{\partial x_k}(x,y) \frac{\partial}{\partial y}(yp(x-z,y)) \, dy dx \right) \widehat{\phi}(z) \, dz$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty \frac{\partial p}{\partial x_k}(x,y) \frac{\partial}{\partial y} \left(\int_{\mathbb{R}^d} yp(x-z,y)e^{-2\pi i \xi z} \, dz \right) \phi(\xi) \, dy dx d\xi$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty \frac{\partial p}{\partial x_k}(x,y) \left(e^{-2\pi i \xi x}(1 - 2\pi|\xi|y)e^{-2\pi|\xi|y} \right) \phi(\xi) \, dy dx d\xi$$

$$= \int_{\mathbb{R}^d} \int_0^\infty 2\pi i \xi_k (1 - 2\pi|\xi|y)e^{-4\pi|\xi|y} \phi(\xi) \, dy d\xi$$

$$= \int_{\mathbb{R}^d} \frac{i\xi_k}{4|\xi|} \phi(\xi) \, dy d\xi.$$

The equation (9.10) follows from the fact that the Fourier transform of $c_d \frac{z_k}{|z|^{d+1}}$ is $-i\frac{\xi_k}{|\xi|}$ in the principal sense. Thus we can write (9.2) as

(9.11)

$$\mathbb{K}_{\mathbb{H}^{(k)}}(z) = \left(-4 \int_0^\infty \int_{\mathbb{R}^d} \left(\frac{1}{h} - 1 \right) \frac{\partial p_0(x,y)}{\partial x_k} \frac{\partial}{\partial y}(yp_z(x,y)) \, dx dy \right) \mathbb{1}_{\{|z|\geq 1\}}(z) + c_d \frac{z_k}{|z|^{d+1}}.$$

Since the Fourier transform of the second term is $-i\frac{\xi_k}{|\xi|}$ (the Fourier transform of the classical Riesz transforms), it is enough to show that

$$\int_{\{|z|\geq 1\}} J_1^{(k)}(z) e^{-2\pi i z \cdot \xi} \, dz, \quad \int_{\{|z|\geq 1\}} J_2^{(k)}(z) e^{-2\pi i z \cdot \xi} \, dz$$

are uniformly bounded in ξ, where

(9.12) $$J_1^{(k)}(z) := \int_0^\infty \int_{\mathbb{R}^d} \left(\frac{1}{h(x,y)} - 1 \right) \frac{x_k y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}}(|x-z|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy,$$

(9.13) $$J_2^{(k)}(z) := \int_0^\infty \int_{\mathbb{R}^d} \left(\frac{1}{h(x,y)} - 1 \right) \frac{x_k y^4}{(|x|^2 + y^2)^{\frac{d+3}{2}}(|x-z|^2 + y^2)^{\frac{d+3}{2}}} \, dx dy.$$

By the estimate (4.4), $|\frac{1}{h(x,y)} - 1| \leq C_d \frac{\sqrt{x^2 + y^2}}{y}$, and

(9.14) $$\int_{\{|x|\geq \frac{1}{2}\}} (|x|^2 + y^2)^{-\frac{d+1}{2}} \, dx \leq \int_{\{|x|\geq \frac{1}{2}\}} |x|^{-(d+1)} \, dx \leq C_d,$$

for $0 \leq y \leq 1$ we have

$$\int_{\{|z|\geq 1\}} \int_0^1 \int_{\{|x|\leq \frac{1}{2}\}} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k|y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}}(|x-z|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy dz$$

$$\leq \int_0^1 \int_{\{|x|\leq \frac{1}{2}\}} \left(\int_{\{|x-z|\geq \frac{1}{2}\}} \frac{1}{(|x-z|^2 + y^2)^{\frac{d+1}{2}}} \, dz \right) \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k|y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}}} \, dx dy$$

$$\leq C_d \int_0^1 \int_{\{|x|\leq \frac{1}{2}\}} \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}}} \, dx dy \leq C_d.$$

Note that in the first inequality, we used the fact that if $|x| \leq \frac{1}{2}$ and $|z| \geq 1$, then $|x - z| \geq \frac{1}{2}$. Similarly,

$$\int_{\{|z| \geq 1\}} \int_1^\infty \int_{\{|x| \leq \frac{1}{2}\}} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}} (|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dx \, dy \, dz$$

$$\leq C_d \int_1^\infty \left(\int_{\{|x| \geq \frac{1}{2}\}} \frac{1}{(|x|^2 + y^2)^{\frac{d+1}{2}}} \, dx \right) \left(\int_{\{|x| \leq \frac{1}{2}\}} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}}} \, dx \right) dy$$

$$\leq C_d \int_1^\infty \int_{\{|x| \leq \frac{1}{2}\}} y^{-d-1} \, dx \, dy \leq C_d.$$

On the other hand, it follows from (4.4), (9.14), and the bound

$$(9.15) \qquad\qquad \int_{\{|z| \geq 1\}} \frac{y}{(|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dz \leq C_d$$

that

$$\int_{\{|z| \geq 1\}} \int_0^1 \int_{\{|x| \geq \frac{1}{2}\}} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}} (|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dx \, dy \, dz$$

$$= \int_0^1 \int_{\{|x| \geq \frac{1}{2}\}} \left(\int_{\{|z| \geq 1\}} \frac{y}{(|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dz \right) \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y}{(|x|^2 + y^2)^{\frac{d+3}{2}}} \, dx \, dy$$

$$\leq C_d \int_0^1 \int_{\{|x| \geq \frac{1}{2}\}} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y}{(|x|^2 + y^2)^{\frac{d+3}{2}}} \, dx \, dy$$

$$\leq C_d \int_0^1 \int_{\{|x| \geq \frac{1}{2}\}} \frac{1}{(|x|^2 + y^2)^{\frac{d+1}{2}}} \, dx \, dy \leq C_d.$$

By Lemma 4.1, we know that $|1/h(x,y) - 1| \leq \frac{C_d}{y}$ for $y \geq 1$. Using this,

$$\int_{\{|z| \geq 1\}} \int_1^\infty \int_{\{|x| \geq \frac{1}{2}\}} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y^2}{(|x|^2 + y^2)^{\frac{d+3}{2}} (|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dx \, dy \, dz$$

$$= \int_1^\infty \int_{\{|x| \geq \frac{1}{2}\}} \left(\int_{\{|z| \geq 1\}} \frac{y}{(|x - z|^2 + y^2)^{\frac{d+1}{2}}} \, dz \right) \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y}{(|x|^2 + y^2)^{\frac{d+3}{2}}} \, dx \, dy$$

$$\leq C_d \int_1^\infty \int_{\{|x| \geq \frac{1}{2}\}} \frac{1}{(|x|^2 + y^2)^{\frac{d+2}{2}}} \, dx \, dy$$

$$\leq C_d \int_1^\infty \frac{1}{y^2} \, dy \int_{\mathbb{R}^d} \frac{1}{(|w|^2 + 1)^{\frac{d+2}{2}}} \, dw \leq C_d.$$

In the last inequality, we have used the change of variable $wy = x$. Thus, we get

$$(9.16) \qquad\qquad \left| \int_{\{|z| \geq 1\}} J_1^{(k)}(z) e^{-2\pi i z \cdot \xi} \, dz \right| \leq \int_{\{|z| \geq 1\}} |J_1^{(k)}(z)| \, dz \leq C_d.$$

Using the trivial bound $y^2/(|x-z|^2+y^2) \leq 1$, it follows from the previous argument that

$$\left| \int_{\{|z|\geq 1\}} J_2^{(k)}(z) e^{-2\pi i z \cdot \xi} \, dz \right|$$

$$\leq \int_{\{|z|\geq 1\}} \int_0^\infty \int_{\mathbb{R}^d} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y^4}{(|x|^2+y^2)^{\frac{d+3}{2}} (|x-z|^2+y^2)^{\frac{d+3}{2}}} \, dx dy dz$$

$$\leq \int_{\{|z|\geq 1\}} \int_0^\infty \int_{\mathbb{R}^d} \left| \frac{1}{h(x,y)} - 1 \right| \frac{|x_k| y^2}{(|x|^2+y^2)^{\frac{d+3}{2}} (|x-z|^2+y^2)^{\frac{d+1}{2}}} \, dx dy dz$$

$$\leq C_d.$$

$\square$

Remark 9.2. *Note that the proof of the boundedness of the Fourier transform for $d=1$ shows that in fact the function*

$$J(z) = \frac{1}{\pi z} \left(\int_0^\infty \frac{2y^3}{(y^2+\pi^2 z^2)\sinh^2(y)} \, dy \right) \mathbb{1}_{\{|z|\geq 1\}}(z)$$

is in $L^1(\mathbb{R})$ with $\|J\|_{L^1} \approx 0.09956$. Similarly, for $d>1$, the proof shows that $J_1^{(k)}(z)\mathbb{1}_{\{|z|\geq 1\}}(z)$ and $J_2^{(k)}(z)\mathbb{1}_{\{|z|\geq 1\}}(z)$ are in $L^1(\mathbb{R}^d)$ with bounds depending only on d.

This gives the following

Corollary 9.3. *For $d=1$, the continuous probabilistic Hilbert transform is given by*

$$\mathbb{K}_{\mathbb{H}}(z) = \frac{1}{\pi z} \left(1 + \int_0^\infty \frac{2y^3}{(y^2+\pi^2 z^2)\sinh^2(y)} \, dy \right) \mathbb{1}_{\{|z|\geq 1\}}(z) + \frac{1}{\pi z} \mathbb{1}_{\{|z|<1\}}(z)$$

$$(9.17) \qquad = J(z) + \frac{1}{\pi z},$$

where $\|J\|_{L^1} \approx 0.09956$.
Similarly for $d>1$,

$$(9.18) \qquad\qquad\qquad \mathbb{K}_{\mathbb{H}^{(k)}}(z) = J_1^{(k)} + J_2^{(k)} + c_d \frac{z_k}{|z|^{d+1}},$$

$$= J^{(k)} + c_d \frac{z_k}{|z|^{d+1}},$$

where $\|J^{(k)}\|_{L^1} \leq C_d$ where C_d depends only on d.

With J and $J^{(k)}$ as above, the probabilistic continuous Hilbert and Riesz transforms are of the form:

$$(9.19) \qquad\qquad Tf = \mathbb{K}_{\mathbb{H}} * f = J * f + Hf,$$

$$(9.20) \qquad T^k f = \mathbb{K}_{\mathbb{H}^{(k)}} * f = J^{(k)} * f + R^{(k)} * f, \quad k = 1, \dots, d.$$

Recall that for $\varepsilon > 0$, $1 < p < \infty$ and a function F on $\mathbb{R}^d$, denote $\tau_\varepsilon F(x) = \varepsilon^{d/p} F(\varepsilon x)$. We define $(T^k)^\varepsilon = \tau_{1/\varepsilon} T^k \tau_\varepsilon$. Following the proof of Lemma 7.7, one has the following

Lemma 9.4. *Suppose that F is a smooth and compactly supported function on $\mathbb{R}^d$. Then*

$$\lim_{\varepsilon \to 0^+} (T^k)^\varepsilon F(x) = R^{(k)}F(x),$$

for every $x \in \mathbb{R}^d$. As a result, we have $\cot(\pi/(2p^*)) = \|R^{(k)}F\|_{p \to p} \leq \|T^k\|_{p \to p}$. The same holds for T as well.

Proof. Note that we have

$$(T^k)^\varepsilon F(x) = \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{K}_{\mathbb{H}^{(k)}}(y)(F(x - \varepsilon y) - F(x + \varepsilon y))\, dy$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} K^{(k)}(y)(F(x - \varepsilon y) - F(x + \varepsilon y))\, dy$$

$$+ \frac{1}{2} \int_{|y| \geq 1} (\mathbb{K}_{\mathbb{H}^{(k)}}(y) - K^{(k)}(y))(F(x - \varepsilon y) - F(x + \varepsilon y))\, dy.$$

Since $K^{(k)}(y) = \varepsilon^d K^{(k)}(\varepsilon y) = -K^{(k)}(-y)$, we have

$$\frac{1}{2} \int K^{(k)}(y)(F(x - \varepsilon y) - F(x + \varepsilon y))\, dy = R^{(k)}F(x).$$

Since F is smooth and compactly supported, there exists $R > 0$ such that $|F(x - \varepsilon y) - F(x + \varepsilon y)| \leq C_1 \mathbb{1}_{B(0,R)}(\varepsilon y)|\varepsilon y|$ for some $C_1 > 0$. By Lemma 7.6, for given $\delta > 0$, there exists $r > 0$ such that $|\mathbb{K}_{\mathbb{H}^{(k)}}(y) - K^{(k)}(y)| \leq \delta |y|^{-d}$ for $|y| > r$. Let $C_2 = \sup_{|y| \geq 1} |\mathbb{K}_{\mathbb{H}^{(k)}}(y) - K^{(k)}(y)|$, then

$$\int_{|y| \geq 1} (\mathbb{K}_{\mathbb{H}^{(k)}}(y) - K^{(k)}(y))(F(x - \varepsilon y) - F(x + \varepsilon y))\, dy$$

$$\leq C_1 \varepsilon \int_{1 \leq |y| \leq R/\varepsilon} |y||\mathbb{K}_{\mathbb{H}^{(k)}}(y) - K^{(k)}(y)|\, dy$$

$$\leq C_1 C_2 \varepsilon \int_{1 \leq |y| \leq r} |y|\, dy + C_1 \delta \varepsilon \int_{1 \leq |y| \leq R/\varepsilon} |y|^{1-d}\, dy$$

$$\leq C_3 r^2 \varepsilon + C_4 \delta.$$

Choose δ small enough and letting $\varepsilon \to 0$, we conclude that $(T^k)^\varepsilon(F)(x)$ converges to $R^{(k)}(F)(x)$. By Fatou's lemma, we also have the lower bound of the L^p norm of T^k as desired. $\qquad\square$

We record the L^p-boundedness of the operators in the following Theorem.

Theorem 9.5. *For $1 < p < \infty$,*

$$(9.21) \quad \cot(\pi/(2p^*)) \leq \|T\|_{L^p \to L^p} \leq \|J\|_{L^1} + \cot(\pi/(2p^*)) \approx 0.09956 + \cot(\pi/(2p^*))$$

and

$$(9.22) \quad \cot(\pi/(2p^*)) \leq \|T^k\|_{L^p \to L^p} \leq \|J^{(k)}\|_{L^1} + \cot(\pi/(2p^*)) \leq C_d + \cot(\pi/(2p^*)),$$

where C_d depends on the dimension d.

Conjecture 9.6. *We conjecture that the L^p-norm of the operator T should be $\cot(\pi/(2p^*))$ and similarly for T^k. For the latter, even the conjecture that the norm is independent of d would be of interest. See the discussion preceding (6.13).*

Remark 9.7. *Recall that the smoothness condition $|\nabla K(z)| \leq \kappa |z|^{-(d+1)}$ can be relaxed with Hörmander's condition*

$$(9.23) \qquad \sup_{y \neq 0} \int_{|x| \geq 2|y|} |K(x-y) - K(x)|\, dx \leq B < \infty,$$

see [37, *Theorem 4.3.3*] *and* [65, *Corollary on p.34, Theroem 2, p.35*]. *In particular, if K satisfies $|K(x)| \leq \kappa |x|^{-d}$ and Hörmander's condition, then the convolution operator T with kernel K is bounded on L^p, $1 < p < \infty$.*

Theorem 9.8. *The kernels $\mathbb{K}_{\mathbb{H}(k)}$ satisfy the Hörmander's condition.*

Proof. We have already seen that

$$|\nabla \mathbb{K}_{\mathbb{H}(k)}(x)| \leq \frac{C}{|x|^{d+1}}, \quad \text{for } |x| \neq 0, 1.$$

Suppose $|y| > 1$. If $|x| \geq 2|y|$ then

$$|\theta(x-y) + (1-\theta)x| = |x - \theta y| \geq (1 - \tfrac{\theta}{2})|x| \geq (2 - \theta)|y|$$

for all $0 \leq \theta \leq 1$. By Taylor's theorem, we have

$$|\mathbb{K}_{\mathbb{H}(k)}(x-y) - \mathbb{K}_{\mathbb{H}(k)}(x)| \leq |\nabla \mathbb{K}_{\mathbb{H}(k)}(x - \theta y)||y| \leq \frac{C|y|}{|x - \theta y|^{d+1}} \leq \frac{C|y|}{|x|^{d+1}},$$

which leads to

$$\int_{|x| \geq 2|y|} |\mathbb{K}_{\mathbb{H}(k)}(x-y) - \mathbb{K}_{\mathbb{H}(k)}(x)|\, dx \leq C|y| \int_{2|y|}^{\infty} \frac{1}{r^2}\, dr \leq C < \infty.$$

If $|y| \leq 1$ and $|x| > |y| + 1$, then $|x - \theta y| > 1$ for $0 \leq \theta \leq 1$. Thus, the same argument yields

$$\int_{|x| \geq |y|+1} |\mathbb{K}_{\mathbb{H}(k)}(x-y) - \mathbb{K}_{\mathbb{H}(k)}(x)|\, dx \leq C < \infty.$$

Let $\frac{1}{4} < |y| \leq 1$ and $2|y| \leq |x| \leq |y| + 1$. Using $|\mathbb{K}_{\mathbb{H}(k)}(x)| \leq \kappa |x|^{-d}$ and $|x - y| \geq \frac{1}{2}|x|$, we get

$$\int_{2|y| \leq |x| \leq |y|+1} |\mathbb{K}_{\mathbb{H}(k)}(x-y) - \mathbb{K}_{\mathbb{H}(k)}(x)|\, dx \leq C \int_{2|y| \leq |x| \leq |y|+1} |x|^{-d}\, dx = C \left| \log\left(\frac{|y|+1}{2|y|} \right) \right|,$$

which is bounded for $|y| \in (\frac{1}{4}, 1]$. Suppose $|y| \leq \frac{1}{4}$ and $\frac{3}{4} \leq |x| \leq |y| + 1$, then the same argument gives

$$\int_{\frac{3}{4} \leq |x| \leq |y|+1} |\mathbb{K}_{\mathbb{H}(k)}(x-y) - \mathbb{K}_{\mathbb{H}(k)}(x)|\, dx \leq C \int_{\frac{3}{4} \leq |x| \leq |y|+1} |x|^{-d}\, dx = C|\log(\tfrac{4}{3}(|y|+1))| < \infty.$$

If $|y| \leq \frac{1}{4}$ and $2|y| \leq |x| < \frac{3}{4}$, then $|x - \theta y| < 1$. Thus it follows from the gradient bound that

$$\int_{2|y| \leq |x| < \frac{3}{4}} |\mathbb{K}_{\mathbb{H}(k)}(x-y) - \mathbb{K}_{\mathbb{H}(k)}(x)|\, dx \leq C|y| \int_{2|y|}^{\frac{3}{4}} \frac{1}{r^2}\, dr \leq C < \infty.$$

Therefore the kernel $\mathbb{K}_{\mathbb{H}(k)}$ satisfies Hörmander's condition and the $L^p(\mathbb{R}^d)$-boundedness of the operators for $1 < p < \infty$ also follows from the Calderón–Zygmund theory. $\qquad \square$

10. A method of rotations for discrete Riesz trasforms

Given the fact that the classical method of rotations can be used to show that the Riesz transforms (and other singular integrals) in $\mathbb{R}^d$ have norms bounded above by the norm of the Hilbert transform, as discussed in Section 6.2, it is natural to ask if there is a discrete version of such a technique that would reduce the boundedness of operators on $\ell^p(\mathbb{Z}^d)$ (with some assumptions on their kernel) to the boundedness of H_{dis} on $\ell^p(\mathbb{Z})$. While this does not seem to be the case for the setting of the discrete Riesz transform as defined in (1.14), we can define closely related operators for which such a procedure is possible.

10.1. **Two-dimensional case.** We first consider the $d = 2$ case where a particularly simple expression for the discrete transform is available. For $j = 1, 2$, from (1.12) we have

$$R^{(i)}f(x) = \frac{1}{2\pi} \, \mathrm{p.v.} \int_{\mathbb{R}^2} f(y) \frac{x_i - y_i}{|x - y|^3} \, dy.$$

Note that

$$\frac{\partial^2 |y|}{\partial y_j^2} = \frac{1}{|y|} - \frac{y_j^2}{|y|^3} = \frac{|y|^2 - y_j^2}{|y|^3} = \frac{y_i^2}{|y|^3},$$

where $i = 1$, if $j = 2$ and $i = 2$, if $j = 1$. Hence, the kernel of $R^{(i)}$ is given by

$$\frac{1}{y_i} \frac{\partial^2 |y|}{\partial y_j^2}.$$

Although not necessarily natural, this motivates the following definition for a different variant of discrete Riesz transforms

$$\mathcal{R}^{(i)}f(n) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}^2} f(m) \frac{|n - m + e_j| + |n - m - e_j| - 2|n - m|}{n_i - m_i} \mathbb{1}_{\{m_i \neq n_i\}}.$$

For simplicity, we consider $i = 1$. Fix $a, b \in \mathbb{R}$ and define the directional discrete Hilbert transform via the formula

$$\mathcal{H}_{a,b}f(n) = \frac{1}{\pi} \sum_{m \in \mathbb{Z}^2} \frac{f(m)}{n_1 - m_1} \mathbb{1}_{\{n_1 \neq m_1, n_2 - \lfloor an_1 + b \rfloor = m_2 - \lfloor am_1 + b \rfloor\}}.$$

The intuition behind this definition is as follows. We split $\mathbb{Z}^2$ into an infinite family of "one-dimensional" sets

$$F_{a,b,l} = \{(k, \lfloor ak + b \rfloor + l) : k \in \mathbb{Z}\} = \{n \in \mathbb{Z}^2 : n_2 - \lfloor an_1 + b \rfloor = l\},$$

where l takes arbitrary integer values. Then $\mathcal{H}_{a,b}$ acts as a (one-dimensional) discrete Hilbert transform on each of the fibers $F_{a,b,l}$. In particular, the above interpretation combined with Corollary 8.2 immediately gives that

$$\|\mathcal{H}_{a,b}\|_{p \to p} = \cot(\tfrac{\pi}{2p^*}),$$

which is the norm of the continuous Hilbert transform.

Theorem 10.1. *For compactly supported $f : \mathbb{Z}^2 \to \mathbb{R}$, we have*

(10.1) $$\mathcal{R}^{(1)}f(n) = \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \frac{1}{(1 + a^2)^{3/2}} \mathcal{H}_{a,b}f(n) \, da \, db,$$

Proof. Formula (10.1) is equivalent to

$$\frac{1}{2\pi} \frac{|n-m+e_2| + |n-m-e_2| - 2|n-m|}{n_1 - m_1}$$

$$= \frac{1}{2} \int_0^1 \int_{-\infty}^\infty \frac{1}{(1+a^2)^{3/2}} \frac{1}{\pi} \frac{1}{m_1 - n_1} \mathbb{1}_{\{m_2 - \lfloor am_1 + b\rfloor = n_2 - \lfloor an_1 + b\rfloor\}} \, dadb,$$

whenever $m_1 \neq n_1$. After elementary simplification, we need to prove that

$$|n-m+e_2| + |n-m-e_2| - 2|n-m|$$

$$= \int_0^1 \int_{-\infty}^\infty \frac{1}{(1+a^2)^{3/2}} \mathbb{1}_{\{m_2 - \lfloor am_1 + b\rfloor = n_2 - \lfloor an_1 + b\rfloor\}} \, dadb.$$

We denote the right-hand side of the above equality by I.

The integrand in I is a periodic function of b, with period 1. Therefore, we may integrate with respect to b over an arbitrary interval of unit length. For convenience, we choose this to be $[-an_1, -an_1 + 1)$, so that $\lfloor an_1 + b\rfloor = 0$, and we substitute $c = an_1 + b$. It follows that

$$I = \int_{-\infty}^\infty \int_{-an_1}^{-an_1+1} \frac{1}{(1+a^2)^{3/2}} \mathbb{1}_{\{m_2 - \lfloor am_1 + b\rfloor = n_2\}} \, dbda$$

$$= \int_{-\infty}^\infty \int_0^1 \frac{1}{(1+a^2)^{3/2}} \mathbb{1}_{\{m_2 - \lfloor a(m_1 - n_1) + c\rfloor = n_2\}} \, dcda.$$

We consider the case $m_1 > n_1$, the remaining case $m_1 < n_1$ being very similar. We have

$$I = \int_0^1 \int_{-\infty}^\infty \frac{1}{(1+a^2)^{3/2}} \mathbb{1}_{\{m_2 - \lfloor a(m_1 - n_1) + c\rfloor = n_2\}} \, dadc$$

$$= \int_0^1 \int_{(m_2 - n_2 - c)/(m_1 - n_1)}^{(m_2 - n_2 - c + 1)/(m_1 - n_1)} \frac{1}{(1+a^2)^{3/2}} \, dadc$$

$$= \int_0^1 \left(\frac{m_2 - n_2 - c + 1}{\sqrt{(m_2 - n_2 - c + 1)^2 + (m_1 - n_1)^2}} - \frac{m_2 - n_2 - c}{\sqrt{(m_2 - n_2 - c)^2 + (m_1 - n_1)^2}} \right) dc$$

$$= \left(-\sqrt{(m_2 - n_2)^2 + (m_1 - n_1)^2} + \sqrt{(m_2 - n_2 - 1)^2 + (m_1 - n_1)^2} \right)$$

$$\quad - \left(-\sqrt{(m_2 - n_2 + 1)^2 + (m_1 - n_1)^2} + \sqrt{(m_2 - n_2)^2 + (m_1 - n_1)^2} \right)$$

$$= |m - n - e_2| + |m - n + e_2| - 2|m - n|,$$

as desired. $\qquad\square$

Note that

$$\int_{-\infty}^\infty \frac{1}{(1+a^2)^{3/2}} \, da = 2.$$

This, as in the classical method of rotations, immediately leads to the following estimate.

Corollary 10.2. *We have*

$$\|\mathcal{R}^{(1)}\|_{\ell^p \to \ell^p} \leq \cot(\tfrac{\pi}{2p^*}).$$

On the other hand, we have the following perfect analogue of Lemma 7.6 which gives the opposite inequality.

Lemma 10.3. *If we denote by $K_{\mathcal{R}^{(k)}}(n)$ the kernel of $\mathcal{R}^{(k)}$, then*

$$\lim_{|n|\to\infty} |n|^2 |K_{\mathcal{R}^{(k)}}(n) - K_{R_{\text{dis}}^{(k)}}(n)| = 0.$$

Proof. The argument boils down to an application of Taylor's theorem and elementary estimates. If $n = (n_1, n_2) \in \mathbb{Z}^2$, $|n| \geq 2$ and $n_1 \neq 0$, then

$$|n - e_2| + |n + e_2| - 2|n| = \sqrt{n_1^2 + (n_2 - 1)^2} + \sqrt{n_1^2 + (n_2 + 1)^2} - 2\sqrt{n_1^2 + n_2^2}$$

$$= \int_{-1}^{1} (1 - |x|) \frac{n_1^2}{(n_1^2 + (n_2 - x)^2)^{3/2}} \, dx,$$

and hence

$$\left| \frac{|n - e_2| + |n + e_2| - 2|n|}{n_1} - \frac{n_1}{(n_1^2 + n_2^2)^{3/2}} \right|$$

$$= \left| \int_{-1}^{1} (1 - |x|) \frac{n_1}{(n_1^2 + (n_2 + x)^2)^{3/2}} \, dx - \int_{-1}^{1} (1 - |x|) \frac{n_1}{(n_1^2 + n_2^2)^{3/2}} \, dx \right|$$

$$\leq \int_{-1}^{1} (1 - |x|) \left| \frac{n_1}{(n_1^2 + (n_2 + x)^2)^{3/2}} - \frac{n_1}{(n_1^2 + n_2^2)^{3/2}} \right| dx.$$

However, $|n + ye_2| \geq |n| - |y| \geq |n| - 1 \geq \frac{1}{2}|n|$ when $|y| \leq 1$ and $|n| \geq 2$, so that

$$\left| \frac{n_1}{(n_1^2 + (n_2 + x)^2)^{3/2}} - \frac{n_1}{(n_1^2 + n_2^2)^{3/2}} \right| = \left| \int_0^x \frac{3n_1(n_2 + y)}{(n_1^2 + (n_2 + y)^2)^{5/2}} \, dy \right|$$

$$\leq \frac{96|n_1|(|n_2| + 1)}{|n|^5} \leq \frac{192}{|n|^3}$$

when $|x| \leq 1$ and $|n| \geq 2$. It follows that

$$\left| \frac{|n - e_2| + |n + e_2| - 2|n|}{n_1} - \frac{n_1}{n_1^2 + n_2^2} \right| \leq \frac{192}{|n|^3} \int_{-1}^{1} (1 - |x|) \, dx = \frac{192}{|n|^3}$$

when $|n| \geq 2$ and $n_1 \neq 0$, and the desired result follows. $\square$

With the above result at hand, we can follow the proof of Lemma 7.7 and show that appropriately rescaled operators $\mathcal{R}^{(k)}$ can be used to approximate (in the point-wise sense) the continuous Riesz transforms $R^{(k)}$, and consequently

$$\|\mathcal{R}^{(k)}\|_{\ell^p \to \ell^p} \geq \|R^{(k)}\|_{L^p \to L^p} = \cot\left(\frac{\pi}{2p^*}\right).$$

We have thus proved the following result.

Theorem 10.4. *The two-dimensional discrete Riesz transforms, defined for $i = 1, 2$ by*

$$\mathcal{R}^{(i)} f(n) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}^2} f(m) \frac{|n - m + e_j| + |n - m - e_j| - 2|n - m|}{n_i - m_i} \mathbb{1}_{\{m_i \neq n_i\}},$$

where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$, have norms on ℓ^p equal to the norms on L^p of the corresponding continuous Riesz transforms: when $1 < p < \infty$, we have

$$\|\mathcal{R}^{(i)}\|_{\ell^p \to \ell^p} = \|R^{(i)}\|_{L^p \to L^p} = \cot(\tfrac{\pi}{2p^*}).$$

10.2. **Higher dimensions.** The same approach works in higher dimensions, too, but a closed-form expression for the corresponding kernel does not seem available. When $d \geq 2$, we define

$$(10.2) \qquad \mathcal{R}^{(k)} f(n) = \sum_{m \in \mathbb{Z}^d} K_{\mathcal{R}^{(k)}}(n - m) f(m),$$

where the kernel for $k = 1$ is given in an integral form as follows. If $n = (n_1, \tilde{n}) \in \mathbb{Z}^d$ with $n_1 \in \mathbb{Z}$ and $\tilde{n} = (n_2, \ldots, n_d) \in \mathbb{Z}^{d-1}$, and if $n_1 > 0$, then

$$K_{\mathcal{R}^{(1)}}(n) = \frac{1}{\pi n_1} \times C_d \int_{[0,1]^{d-1}} \int_{\frac{\tilde{n}-b}{n_1} + [0, \frac{1}{n_1})^{d-1}} \frac{1}{(1 + |a|^2)^{(d+1)/2}} \, da \, db,$$

where C_d is related to the constant c_d in (1.13) via

$$C_d = \left(\int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |a|^2)^{(d+1)/2}} \, da \right)^{-1} = \pi c_d.$$

Furthermore, when $n_1 < 0$, then $K_{\mathcal{R}^{(1)}}(n) = -K_{\mathcal{R}^{(1)}}(-n)$. For a general k, the kernel $K_{\mathcal{R}^{(k)}}(n)$ is equal to $K_{\mathcal{R}^{(1)}}(n')$, where n' is obtained from n by swapping the first and k-th coordinate.

By definition, as in the two-dimensional case, for compactly supported $f : \mathbb{Z}^d \to \mathbb{R}$, we have

$$(10.3) \qquad \mathcal{R}^{(1)} f(n) = C_d \int_{[0,1]^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |a|^2)^{(d+1)/2}} \, \mathcal{H}_{a,b} f(n) \, da \, db,$$

where $\mathcal{H}_{a,b}$ acts as the discrete Hilbert transform with respect to n_1 on each of the fibers

$$F_{a,b,l} = \{(n_1, \lfloor a n_1 + b \rfloor + l) : n_1 \in \mathbb{Z}\} = \{n \in \mathbb{Z}^d : n_j - \lfloor a_j n_1 + b_j \rfloor = l_j, \ j = 2, 3, \ldots, d\},$$

with $l \in \mathbb{Z}^{d-1}$ (here we understand that the floor function in $\lfloor a n_1 + b \rfloor$ acts componentwise). Therefore,

$$\|\mathcal{R}^{(1)}\|_{\ell^p \to \ell^p} \leq \cot(\tfrac{\pi}{2p^*}).$$

On the other hand, below we prove that (as in Lemma 10.3 for $d = 2$)

$$(10.4) \qquad \lim_{|n| \to \infty} |n|^d |K_{\mathcal{R}^{(1)}}(n) - K_{R^{(1)}_{\text{dis}}}(n)| = \lim_{|n| \to \infty} |n|^d \left| K_{\mathcal{R}^{(1)}}(n) - c_d \frac{n_1}{|n|^{(d+1)/2}} \right| = 0.$$

Once this is shown, by the same argument as in the case of $d = 2$, we find that

$$\|\mathcal{R}^{(1)}\|_{\ell^p \to \ell^p} \geq \|R^{(1)}\|_{L^p \to L^p} = \cot(\tfrac{\pi}{2p^*}).$$

Thus, we conclude that in fact the norms are equal. We state this as a theorem.

Theorem 10.5. *The discrete Riesz transforms $\mathcal{R}^{(k)}$ introduced above have norms on ℓ^p equal to the norms on L^p of the corresponding continuous Riesz transforms: when $1 < p < \infty$, we have*

$$\|\mathcal{R}^{(k)}\|_{\ell^p \to \ell^p} = \|R^{(k)}\|_{L^p \to L^p} = \cot(\tfrac{\pi}{2p^*}).$$

Proof. We only need to prove (10.4). As before, we write $n = (n_1, \tilde{n})$, where $\tilde{n} = (n_2, \ldots, n_d)$, and since both kernels are odd functions of n_1, without loss of generality we assume that $n_1 > 0$. We have

$$K_{\mathcal{R}^{(1)}}(n) - K_{R^{(1)}_{\text{dis}}}(n)$$

$$= \frac{c_d}{n_1} \int_{[0,1]^{d-1}} \int_{\frac{\tilde{n}-b}{n_1}+[0,\frac{1}{n_1})^{d-1}} \frac{1}{(1+|a|^2)^{(d+1)/2}} \, da\, db - c_d \frac{n_1}{(n_1^2+|\tilde{n}|^2)^{(d+1)/2}}$$

$$= \frac{c_d}{n_1^d} \int_{[0,1]^{d-1}} \int_{[0,1)^{d-1}} \frac{n_1^{d+1}}{(n_1^2+|\tilde{n}-b+v|^2)^{(d+1)/2}} \, dv\, db - c_d \frac{n_1}{(n_1^2+|\tilde{n}|^2)^{(d+1)/2}}$$

$$= c_d n_1 \int_{[0,1]^{d-1}} \int_{[0,1)^{d-1}} \left(\frac{1}{(n_1^2+|\tilde{n}-b+v|^2)^{(d+1)/2}} - \frac{1}{(n_1^2+|\tilde{n}|^2)^{(d+1)/2}} \right) dv\, db.$$

Since in the given region of integration we have $-b + v \in [-1,1]^{d-1}$, it follows that

$$|K_{\mathcal{R}^{(1)}}(n) - K_{R^{(1)}_{\text{dis}}}(n)| \le c_d n_1 \sup_{w \in [-1,1]^{d-1}} \left| \frac{1}{(n_1^2+|\tilde{n}+w|^2)^{(d+1)/2}} - \frac{1}{(n_1^2+|\tilde{n}|^2)^{(d+1)/2}} \right|.$$

We now simply use the mean value theorem for the function $t \mapsto t^{-d-1}$ evaluated at $t_1 = \sqrt{n_1^2+|\tilde{n}+w|^2}$ and $t_2 = \sqrt{n_1^2+|\tilde{n}|^2} = |n|$: we have

$$\left| \frac{1}{t_1^{d+1}} - \frac{1}{t_2^{d+1}} \right| \le |t_1 - t_2| \times \frac{d+1}{\min\{t_1^{d+2}, t_2^{d+2}\}}.$$

Since $|t_1 - t_2| \le |w| \le \sqrt{d-1}$, we have $t_1 \ge t_2 - \sqrt{d-1} = |n| - \sqrt{d-1} \ge \frac{1}{2}|n|$ when $|n|$ is large enough, and thus

$$\left| \frac{1}{(n_1^2+|\tilde{n}+w|^2)^{(d+1)/2}} - \frac{1}{(n_1^2+|\tilde{n}|^2)^{(d+1)/2}} \right| = \left| \frac{1}{t_1^{d+1}} - \frac{1}{t_2^{d+1}} \right| \le \sqrt{d-1} \times \frac{2^{d+2}(d+1)}{|n|^{d+2}}$$

when $|n|$ is large enough. We thus conclude that when $|n|$ is large enough, then

$$|K_{\mathcal{R}^{(1)}}(n) - K_{R^{(1)}_{\text{dis}}}(n)| \le 2^{d+1}(d+1)c_d \sqrt{d-1} \, \frac{n_1}{|n|^{d+2}}.$$

The right-hand side multiplied by $|n|^d$ goes to zero as $|n| \to \infty$, and the proof is complete. $\square$

We remark that a similar construction of the discrete Riesz transform using the method of rotations can be carried out using the probabilistic discrete Hilbert transform $T_{\mathbb{H}}$ instead of the discrete Hilbert transform H_{dis} applied above. This procedure will lead to a transform with the same norm on ℓ^p, but with a kernel which is greater in absolute value than the kernel of $\mathcal{R}^{(k)}$ (in the point-wise sense). However, we did not pursue this direction.

11. Numerical comparison of kernels

We end with some remarks on numerical comparisons on the kernels for the discrete operators $R^{(k)}_{\text{dis}}$, $\mathcal{R}^{(k)}$, and $T_{\mathbb{H}^{(k)}}$. Numerical evaluation of the kernels for $R^{(k)}_{\text{dis}}$ and $\mathcal{R}^{(k)}$ when $d = 2$ presents no difficulties. The situation is quite different for $T_{\mathbb{H}^{(k)}}$, which is given by a triple integral involving the periodic Poisson kernel $h(x, y)$.
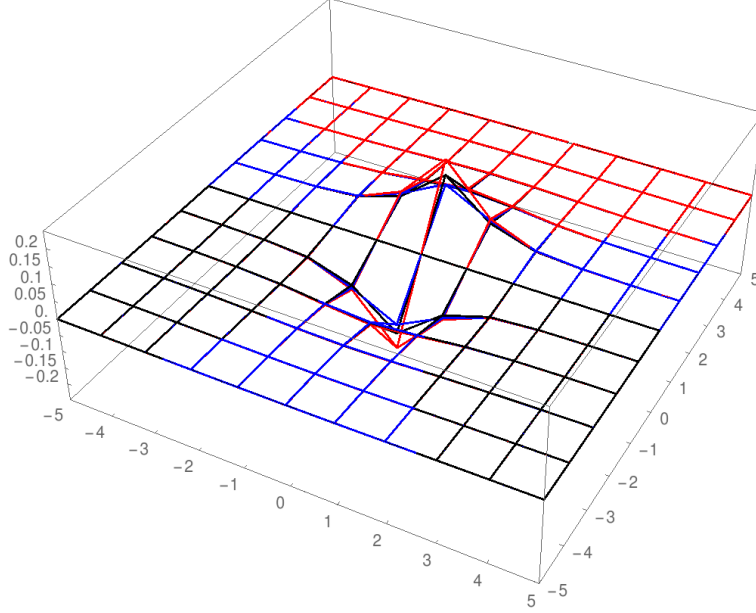
FIGURE 2. Values of the kernels of the three transforms: $\mathbb{K}_{\mathbb{H}^{(1)}}(n_1, n_2)$ (red), $K_{\mathcal{R}^{(1)}}(n_1, n_2)$ (blue), and $K_{R_{\text{dis}}^{(1)}}(n_1, n_2)$ (black) for $n_1, n_2 \in \{-5, -4, \ldots, 5\}$.

In the following numerical simulations we used *Wolfram Mathematica 10* and a relatively naive approach, which may lead to significant errors. That said, the outcome turned out to be relatively stable when we varied the parameters, so we believe that our approximations are correct to roughly fourth significant digit.

The periodic Poisson kernel $h(x, y)$ was approximated using the definition (4.2) when $y \leq \frac{1}{4}$ and using the expression (4.6) based on the Poisson summation formula when $y \geq \frac{1}{4}$. Additionally, since $h(x, y)$ converges to 1 exponentially fast as $y \to \infty$, for $y \geq 10$ we simply approximated $h(x, y)$ by a constant 1. To speed up numerical integration, we evaluated the above numerical approximation to $h(x, y)$ in a limited number of points, and then we used appropriate interpolation to find the values of $h(x, y)$ between these points.

Numerical integration was done using standard methods available in *Mathematica*. Although *Mathematica* warned about slow convergence, the estimated error of numerical integration appears to be less significant than the errors in approximation of the periodic Poisson kernel.

The values of the three kernels are shown in Figure 2. The ratio between the kernels of the probabilistic and the CZ discrete Riesz transform are shown in Figure 3, while a similar plot for the method of rotations and the Riesz transform R_{dis}^1 is shown in Figure 4. Numerical results are presented in Tables 1, 2 and 3.

Our simulations suggest that there is no general point-wise relation between the kernels of $\mathcal{R}^{(k)}$ and $R_{\text{dis}}^{(k)}$, nor there is one between the kernels of $T_{\mathbb{H}^{(k)}}$ and $\mathcal{R}^{(k)}$. However,
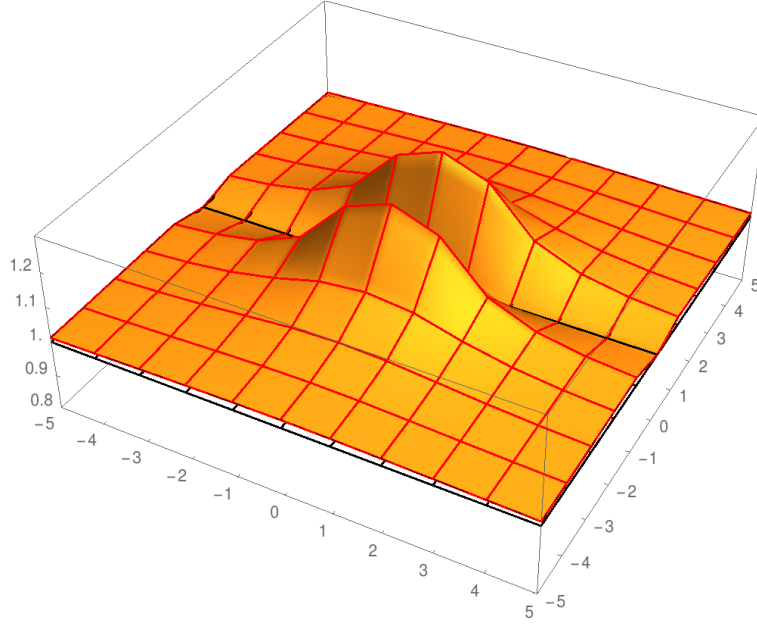
FIGURE 3. Ratio between the kernels of two transforms: $\mathbb{K}_{\mathbb{H}^{(1)}}(n_1, n_2)/K_{R_{\mathrm{dis}}^{(1)}}(n_1, n_2)$ (red) compared with constant 1 (black) for $n_1, n_2 \in \{-5, -4, \dots, 5\}$. When $n_1 = 0$, we set $0/0 = 1$.
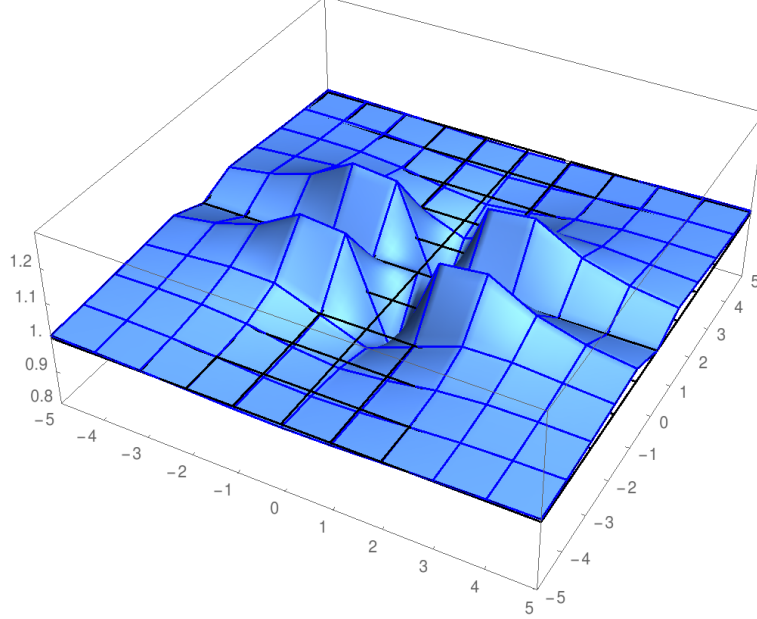

FIGURE 4. Ratio between the kernels of two transforms: $K_{\mathcal{R}^{(1)}}(n_1, n_2)/K_{R_{\mathrm{dis}}^{(1)}}(n_1, n_2)$ (blue) compared with constant 1 (black) for $n_1, n_2 \in \{-5, -4, \dots, 5\}$. When $n_1 = 0$, we set $0/0 = 1$.

TABLE 1. Values of the kernels of the three transforms: $\mathbb{K}_{\mathbb{H}^{(1)}}(n_1, n_2)$ (top row), $K_{\mathcal{R}^{(1)}}(n_1, n_2)$ (middle row) and $K_{R_{\mathrm{dis}}^{(1)}}(n_1, n_2)$ (bottom row) for $n_1, n_2 \in \{0, 1, \ldots, 5\}$, $n_1 \neq 0$. The largest value in each cell is set in bold, while the smallest one is given in slanted type.

		n_2					
		0	1	2	3	4	5
	1	**0.2051**	**0.0698**	0.0158	0.0053	0.0024	0.0012
		0.1318	0.0649	**0.0166**	**0.0055**	**0.0024**	**0.0012**
		0.1592	*0.0563*	*0.0142*	*0.0050*	*0.0023*	*0.0012*
	2	**0.0446**	**0.0315**	**0.0151**	0.0071	0.0037	0.0021
		0.0376	*0.0284*	0.0147	**0.0071**	**0.0037**	**0.0021**
		0.0398	0.0285	*0.0141*	*0.0068*	*0.0036*	*0.0020*
n_1	3	**0.0188**	**0.0160**	**0.0106**	**0.0065**	**0.0039**	**0.0025**
		0.0172	*0.0149*	0.0103	0.0064	0.0039	0.0025
		0.0177	0.0151	*0.0102*	*0.0063*	*0.0038*	*0.0024*
	4	**0.0103**	**0.0094**	**0.0073**	**0.0052**	**0.0036**	**0.0025**
		0.0098	*0.0090*	*0.0071*	0.0051	0.0036	0.0025
		0.0099	0.0091	0.0071	*0.0051*	*0.0035*	*0.0024*
	5	**0.0065**	**0.0061**	**0.0052**	**0.0041**	**0.0031**	**0.0023**
		0.0063	*0.0060*	*0.0051*	0.0040	0.0030	0.0023
		0.0064	0.0060	0.0051	*0.0040*	*0.0030*	*0.0023*

it seems that the kernel of $T_{\mathbb{H}^{(k)}}$ is always greater (in the absolute value) than the kernel of $R_{\mathrm{dis}}^{(k)}$. This leads to the following conjecture which we know is true for $d = 1$ by (1.8).

Conjecture 11.1. *For all $d \geq 2$, we have $|\mathbb{K}_{\mathbb{H}^{(k)}}(n)| \geq |K_{R_{\mathrm{dis}}^{(k)}}(n)|$ for every $n \in \mathbb{Z}^d$.*

The above numerical findings give a little insight into Question 8.7, which asks whether $K_{R_{\mathrm{dis}}^{(k)}}$ is the convolution of $\mathbb{K}_{\mathbb{H}^{(k)}}$ with some probability kernel. Indeed, although intuitively point-wise domination asserted in Conjecture 11.1 appears to be a necessary condition for a positive answer to Question 8.7, neither of these statements implies the other one.

On the other hand, our calculations strongly suggest that in dimension $d = 2$ the maximum of $K_{\mathcal{R}^{(1)}}(n_1, n_2)$ is strictly smaller than the maximum of $K_{R_{\mathrm{dis}}^{(1)}}(n_1, n_2)$; both maxima are attained at $(n_1, n_2) = (1, 0)$. If this is indeed the case, then $K_{R_{\mathrm{dis}}^{(1)}}$ is clearly *not* a convolution of $K_{\mathcal{R}^{(1)}}$ and a probability kernel. Thus, we expect that the analogue of Question 8.7 for $\mathcal{R}^{(k)}$ instead of $T_{\mathbb{H}^{(k)}}$ has a negative answer.

Finally, one can ask if the analogue of Question 8.7 holds for the discrete Riesz transform obtained with the method of rotations, but using the probabilistic discrete Hilbert transform $T_{\mathbb{H}}$ instead of the usual discrete Hilbert transform H_{dis}. We did not attempt to answer this question.

TABLE 2. Values of the ratios $\mathbb{K}_{\mathbb{H}^{(1)}}(n_1, n_2)/K_{R_{\mathrm{dis}}^{(1)}}(n_1, n_2)$ of kernels of two transforms for $n_1, n_2 \in \{0, 1, \ldots, 5\}$, $n_1 \neq 0$.

		n_2					
		0	1	2	3	4	5
	1	**1.2885**	**1.2413**	**1.1127**	**1.0593**	**1.0356**	**1.0235**
	2	**1.1200**	**1.1067**	**1.0717**	**1.0458**	**1.0303**	**1.0211**
n_1	3	**1.0615**	**1.0567**	**1.0450**	**1.0333**	**1.0243**	**1.0180**
	4	**1.0364**	**1.0345**	**1.0298**	**1.0241**	**1.0191**	**1.0150**
	5	**1.0239**	**1.0230**	**1.0208**	**1.0179**	**1.0149**	**1.0123**

TABLE 3. Values of the ratios $K_{\mathcal{R}^{(1)}}(n_1, n_2)/K_{R_{\mathrm{dis}}^{(1)}}(n_1, n_2)$ of kernels of two transforms for $n_1, n_2 \in \{0, 1, \ldots, 5\}$, $n_1 \neq 0$.

		n_2					
		0	1	2	3	4	5
	1	0.8284	**1.1530**	**1.1667**	**1.0947**	**1.0574**	**1.0379**
	2	0.9443	0.9959	**1.0452**	**1.0483**	**1.0385**	**1.0292**
n_1	3	0.9737	0.9873	**1.0094**	**1.0205**	**1.0221**	**1.0199**
	4	0.9848	0.9897	0.9997	**1.0077**	**1.0116**	**1.0125**
	5	0.9902	0.9923	0.9972	**1.0022**	**1.0057**	**1.0075**

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