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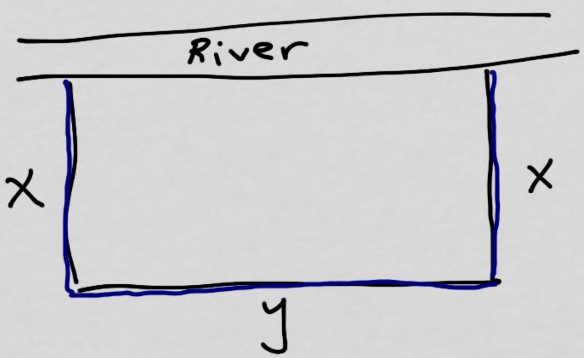
# 4.5 Optimization Problems (part 1)

nothing new - apply methods to finding max/min

example A farm is set up next to a river. Fencing is not required along the river. If 400 m of fencing is available to fence the other 3 sides. Find the dimensions of the farm w/ the largest area.




length and width to max area?

goal: maximize $A = xy$ → objective function (the thing to max/min)

$2x + y = 400$ → constraint (condition that variables must satisfy)

(fencing available

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$A = xy$ → normally, we take derivative, find critical #s and so on

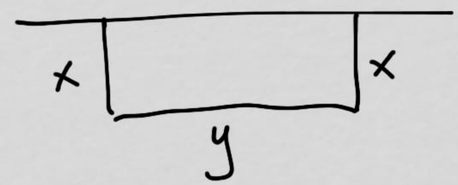
↳ has Two variables

MUST remove one of them by using the constraint

$2x + y = 400 \to y = 400 - 2x$ Sub into the objective

$A = xy = x(400 - 2x)$

$\boxed{A = 400x - 2x^2}$ domain?




$0 \le 2x \le 400$

$\boxed{0 \le x \le 200}$

find critical numbers, then compare A at endpts and at critical numbers

$A' = 400 - 4x$

$A' = 0 \to \boxed{x = 100}$ critical #

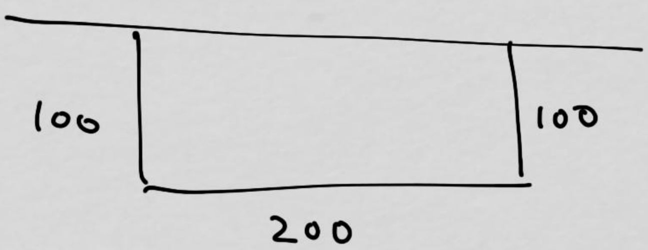
Compare $A = 400x - 2x^2$ at $\underbrace{x = 100,}_{\text{critical \#}}$ $\underbrace{x = 0,\ x = 200}_{\text{endpoints}}$

$A(0) = 0$

$A(100) = 400(100) - 2(100)^2 = 20{,}000$

$A(200) = 400(200) - 2(200)^2 = 0$

So, to maximize $A$, we want $x = 100$, $y = 400 - 2x = 200$



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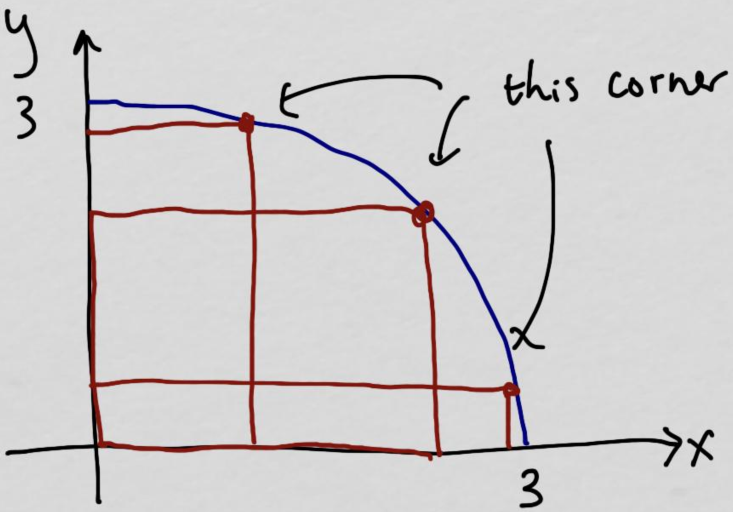
<u>example</u> What is the area of the largest rectangle that can be inscribed inside the first quadrant portion of a circle w/ radius 3?




where to put that corner on the circle to maximize rectangle area?

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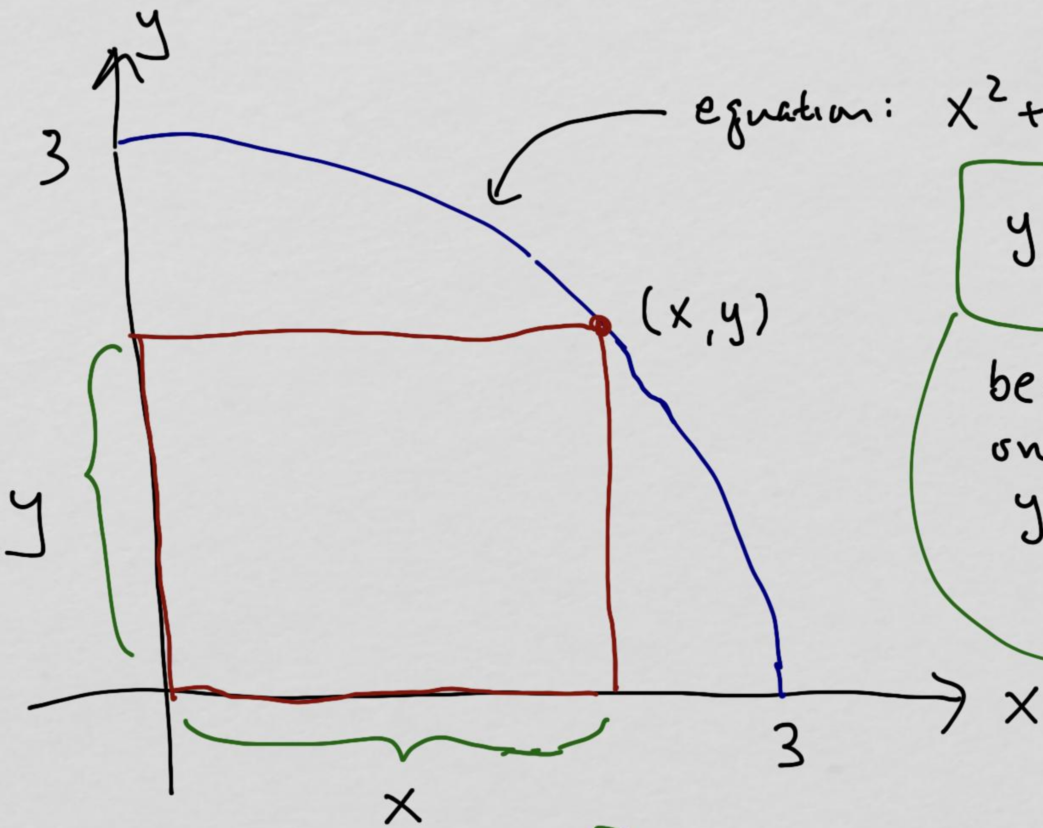



equation: $x^2+y^2=3^2$

$$\boxed{y=\sqrt{9-x^2}}$$

because the corner is on the circle, x and y are always related according to eq. of circle

maximize $A = xy$ → two variables, remove one

$$A=(x)(\sqrt{9-x^2})$$

$$=\sqrt{x^2}\sqrt{9-x^2}=\sqrt{(x^2)(9-x^2)}=\sqrt{9x^2-x^4}=(9x^2-x^4)^{1/2}$$

Domain: $0 \le x \le 3$

$$A = (9x^2 - x^4)^{1/2} \qquad 0 \le x \le 3$$

$$A' = \frac{1}{2}(9x^2 - x^4)^{-1/2}(18x - 4x^3)$$

$$A' = \frac{9x - 2x^3}{(9x^2 - x^4)^{1/2}}$$

$$A' = 0 \rightarrow 9x - 2x^3 = 0$$

$$x(9 - 2x^2) = 0$$

$$\boxed{x = 0}, \quad x^2 = \frac{9}{2} \rightarrow \boxed{x = \frac{\sqrt{3}}{2},} \quad -\cancel{\frac{\sqrt{3}}{2}}$$

$$A' \text{ DNE} \rightarrow (9x^2 - x^4)^{1/2} = 0$$

$$9x^2 - x^4 = 0$$

$$x^2(9 - x^2) = 0 \rightarrow \boxed{x = 0,} \quad \boxed{x = 3,} \quad \cancel{x = -3}$$

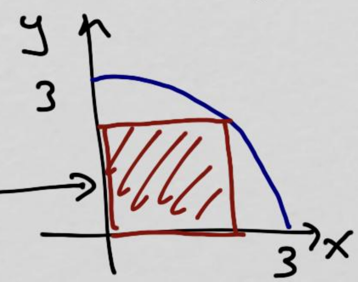
compare:

$$A(0) = 0$$

$$A\left(\frac{\sqrt{3}}{2}\right) = \frac{9}{2}$$

$$A(3) = 0$$

so, the largest such rectangle has area $= \dfrac{9}{2}$



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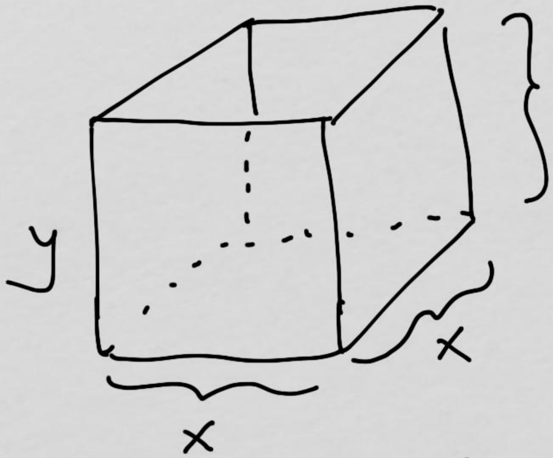
<u>example</u> We want to build a storage box with a square base and a volume of $2\ ft^3$. We don't want a lid for this box. If the material for the base costs $\$3\ /ft^2$ and the material for the sides costs $\$2\ /ft^2$, what dimensions result in the cheapest box?




height is different

base length = base width because

requirement: $x^2 y = 2$ constraint

minimize: $$C = 3\underbrace{(x^2)}_{\text{base area}} + 4 \cdot 2\underbrace{(xy)}_{\text{area of each side}}$$

base costs $\$3/ft^2$

four sides

side costs $\$2/ft^2$

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$C = 3x^2 + 8xy$ has two variables

$x^2 y = 2 \longrightarrow y = \dfrac{2}{x^2}$ sub into $C$

$$C = 3x^2 + 8x\left(\frac{2}{x^2}\right) = 3x^2 + \frac{16}{x}$$

$$C' = 6x - \frac{16}{x^2} = 0 \qquad 6x = \frac{16}{x^2}$$

$$x^3 = \frac{16}{6} = \frac{8}{3} \qquad \boxed{x = \sqrt[3]{\frac{8}{3}}}$$ critical #

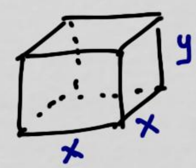
we need to make sure that $x = \sqrt[3]{\frac{8}{3}}$ gives us the minimum $C$

$\rightarrow$ use the First or Second Derivative Test

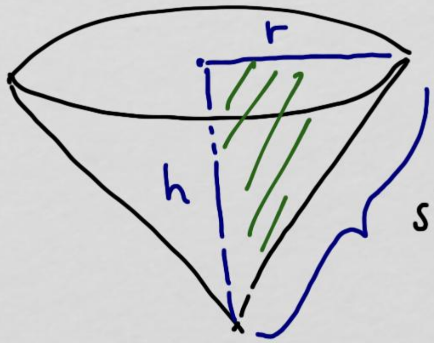
$$C'' = 6 + \frac{32}{x^3}$$

$$C''\left(\sqrt[3]{\tfrac{8}{3}}\right) = 6 + \frac{32}{\left(\sqrt[3]{\frac{8}{3}}\right)^3} > 0 \longrightarrow$$ so, $x = \sqrt[3]{\frac{8}{3}}$ gives a minimum $C$

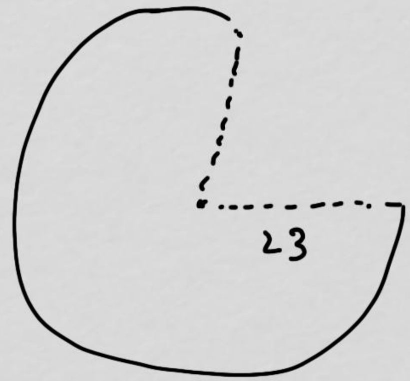
$$y = \frac{2}{x^2} = \frac{2}{\left(\sqrt[3]{\frac{8}{3}}\right)^2}$$



<u>example</u> Find the radius and height of the largest possible cone (max volume) with slant height of 23.




slant height = 23



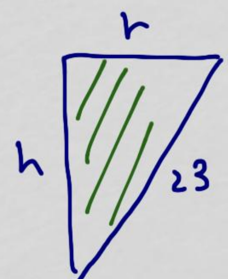

fold into a conical cup

volume of cone

$V = \frac{1}{3}\pi r^2 h$ → two variables $r, h$. $\pi$ is a constant

but we don't have $r$ or $h$

look at a slice of the cone




$r^2 + h^2 = (23)^2 = 529$

$r^2 = 529 - h^2$ replace $r^2$ in $V$ w/ this

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$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(529 - h^2)(h)$$

$$V = \frac{529}{3}\pi h - \frac{1}{3}\pi h^3$$

now find the critical number

$$V' = \frac{529}{3}\pi - \pi h^2 = 0$$

$$\vdots$$

$$h^2 = \frac{529}{3} \qquad h = \sqrt{\frac{529}{3}}$$

verify that this h maximize V

$$V'' = -2\pi h$$

$$V''\left(\sqrt{\frac{529}{3}}\right) < 0 \rightarrow h = \sqrt{\frac{529}{3}}$$ does max V (volume)

$$r^2 = 529 - h^2 \rightarrow r = \sqrt{529 - h^2} = \sqrt{529 - \frac{529}{3}} = \sqrt{\frac{2(529)}{3}}$$