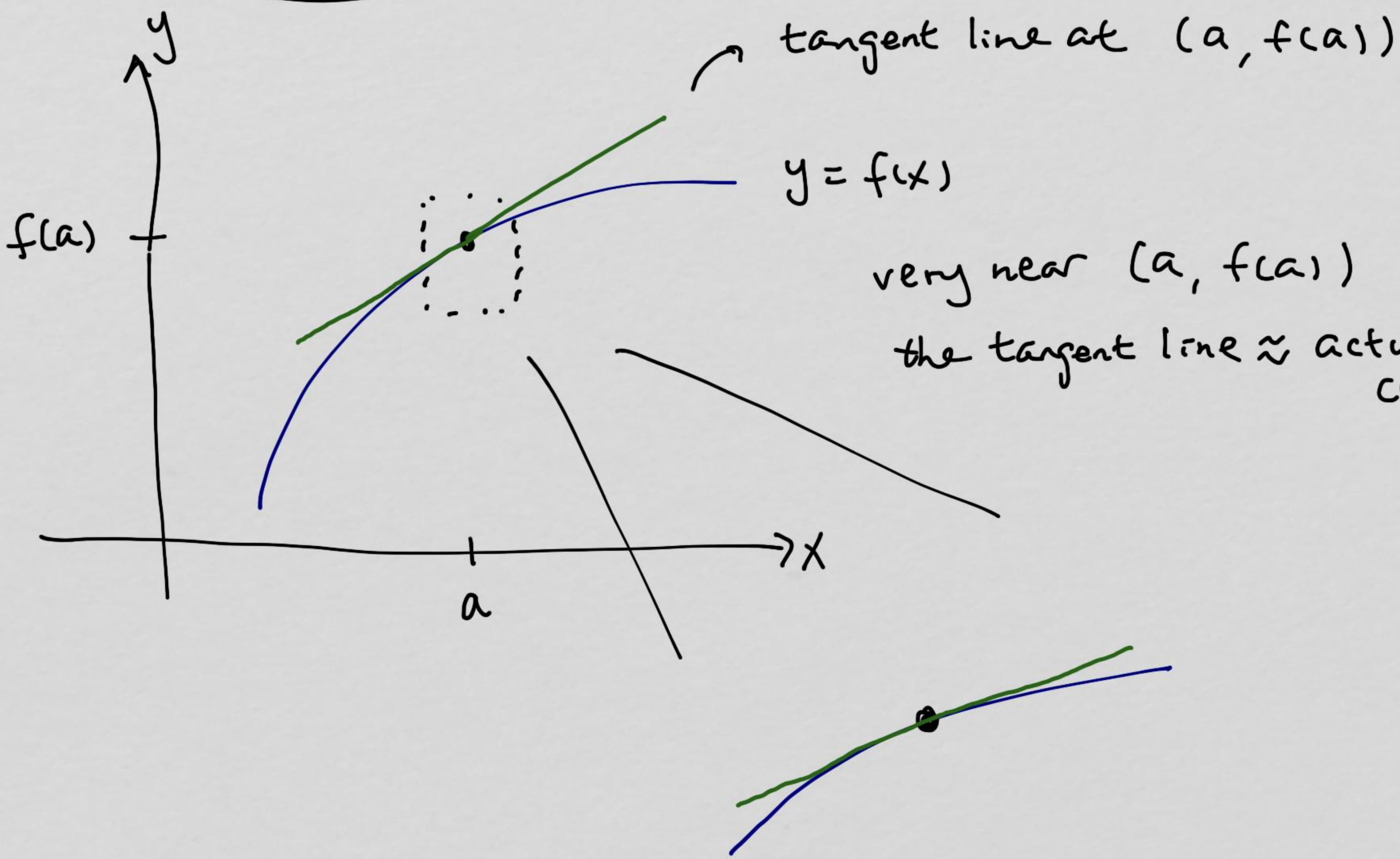
4.6 Linear Approximation and the Differential



very near $(a, f(a))$

the tangent line $\approx$ actual curve

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that means if x is close to a, then we use the equation of the tangent line in place of the actual function $f(x)$

but this is only an estimate — the closer x is to a the better the estimate is

equation of tangent line at $x = a$, $y = f(a)$

slope: $f'(a)$

point: $(a, f(a))$

equation: $y - f(a) = f'(a)(x - a)$

$$\boxed{y = f(a) + f'(a)(x-a) \approx f(x) \text{ if } x \text{ is close to } a}$$

using a tangent line to approximate $f(x)$ is called <u>Linear Approximation of $f(x)$</u>

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$$\boxed{L(x) \approx f(x) = f(a) + \underbrace{f'(a)(x-a)}}$$

this is called the differential

it gives an estimate of the change in $f(x)$ as x changes from a.

example Find a linear approximation of $f(x) = \sin(x)$ $\underbrace{\text{near } x = 0}$

make the tangent line at $x=0 \longrightarrow a = 0$

$$L(x) = f(a) + f'(a)(x-a)$$

$a=0$, $f(a) = \sin(0) = 0$

$f'(x) = \cos(x)$, $f'(a) = \cos(0) = 1$

$$L(x) = 0 + (1)(x-0)$$

$$\boxed{L(x) = x \approx \sin(x)}$$ when x is near 0

becomes close to x

this is another way to explain why $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

use the fact that $\sin(x) \approx x$ near $x=0$ to estimate $\sin(30°)$

change to radians

$$30° = 30 \cdot \frac{\pi}{180} = \frac{\pi}{6}$$

$$\sin(x) \approx x \longrightarrow \boxed{\sin\left(\frac{\pi}{6}\right) \approx \frac{\pi}{6} = 0.52}$$

how close is it to the actual value?

we know $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = 0.5$

pretty close. What is the percentage error?

$$\text{percentage error} = 100 \cdot \frac{|\text{approximation} - \text{true}|}{\text{true}}$$

$$= 100 \cdot \frac{|0.52 - 0.5|}{0.5} = 100 \cdot \frac{0.02}{0.5} = 4$$

So this estimate has a 4% error. (good because we didn't go too far from $x=0$)

try $\sin(90^\circ)$

$90^\circ = \frac{\pi}{2}$

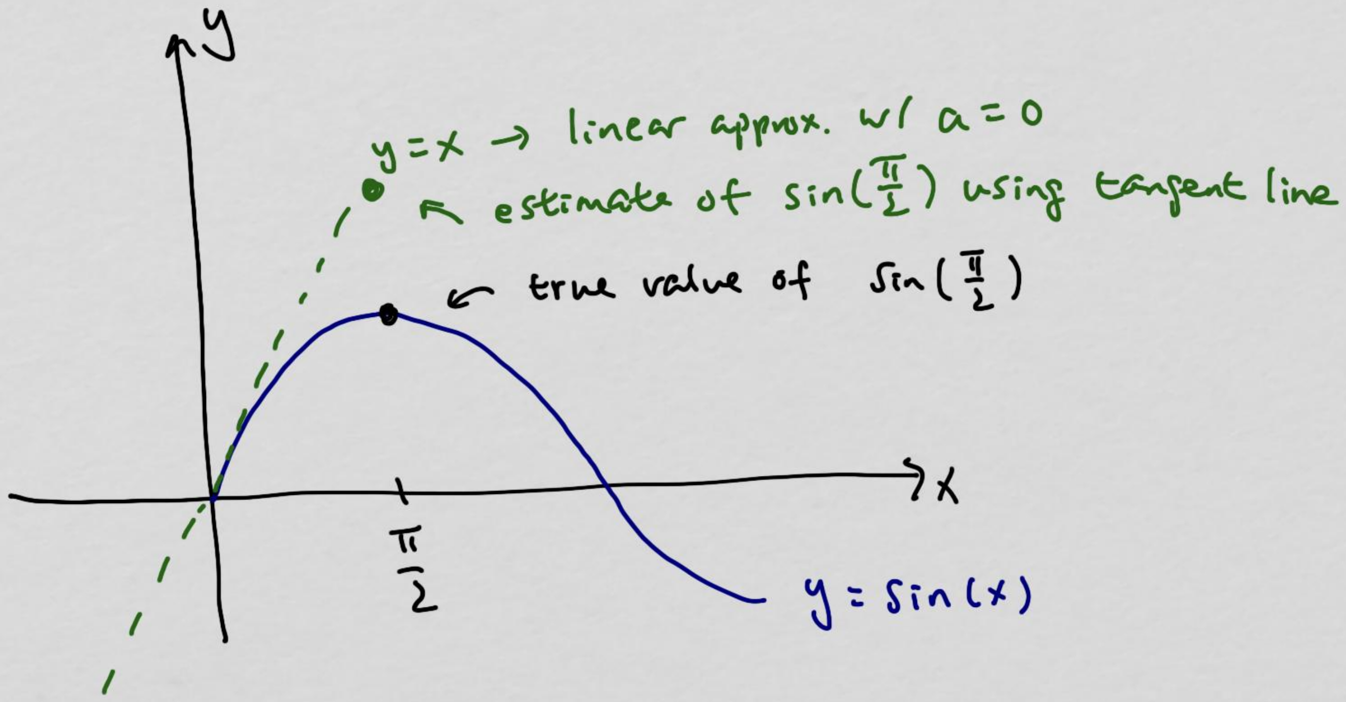
so, $\sin(x) \approx x \longrightarrow \sin\left(\frac{\pi}{2}\right) \approx \frac{\pi}{2}$ linear approx.

≈ 1.57

true value: $\sin\left(\frac{\pi}{2}\right) = 1$

$$\%\ \text{error} = 100 \cdot \frac{|1.57 - 1|}{1} = 57$$

57% error



y
x
y=x → linear approx. w/ a=0
↖ estimate of sin(π/2) using tangent line
← true value of sin(π/2)
π/2
y = sin(x)

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<u>example</u> Use a linear approx. to estimate $\sqrt{108}$

we can view this as $f(x) = \sqrt{x}$ with $x = 108$

(estimate $f(108)$ using a linear approximation)

at what a?

choose a such that it is close to $x = 108$ <u>and</u> $f(a)$ is easy to calculate.

so here, a good choice of a is $a = 100$

because 100 is close to 108 and $f(a) = \sqrt{100}$ is easy to calculate.

now use $L(x) = f(a) + f'(a)(x-a) \approx f(x)$

$$f(a) = \sqrt{100} = 10$$

$$f'(a) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \rightarrow f'(a) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

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$$L(x) = 10 + \frac{1}{2\sqrt{100}}(x-100) \approx f(x) = \sqrt{x}$$

$$\sqrt{108} \approx 10 + \frac{1}{20}(108-100) = 10 + \frac{8}{20} = 10 + \frac{4}{10} = 10.4$$

how good is it?

w/ a calculator, $\sqrt{108} = 10.3923$

note: a is a choice we have to make

we could have used $a = 109$ if we wanted to,

but $f(a) = \sqrt{109}$ is not easy to calculate

(it's as hard as $\sqrt{108}$, which defeats the purpose of using linear approx.)

example Use a linear approximation to estimate $\ln(3)$.

$f(x) = \ln(x)$ with $x = 3$

now find a that is close to $x=3$ and it is easy to calculate $f(a) = \ln(a)$

here, $a = e$ is an excellent choice because $e \approx 2.7$ and $\ln(e) = 1$

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = \ln(e) = 1$

$f'(x) = \dfrac{1}{x}$

$f'(a) = \dfrac{1}{e}$

$L(x) = 1 + \frac{1}{e}(x-e) \approx \ln(x)$ if x is near $a=e$

$$\ln(3) \approx L(3) = 1 + \frac{1}{e}(3-e)$$

$$= 1 + \frac{3}{e} - \frac{e}{e} = \frac{3}{e} \approx 1.1$$

w/ a calculator, $\ln(3) = 1.0986$ very good approximation.