4.7 l'Hospital's Rule

revisit an old problem: $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x^2-4} \to \dfrac{4+2-6}{4-4} \to \dfrac{0}{0}$

indeterminate form of $\dfrac{0}{0}$

old way: $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x^2-4}$

$= \lim\limits_{x\to 2} \dfrac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \boxed{\dfrac{5}{4}}$

$\dfrac{0}{0}$ is actually $\dfrac{\text{small \#}}{\text{small \#}}$

today, we'll use l'Hospital's Rule to handle limits that seem to want to go to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

indeterminate form of $\dfrac{\infty}{\infty}$ $\left(\dfrac{\text{big \#}}{\text{big \#}}\right)$

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l'Hospital's Rule

named after a French noble and mathematician Marquis de l'Hospital

if $\lim_{x \to a} \frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

<u>example</u> $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4} \to \frac{0}{0}$ so we can use l'Hospital's Rule

l'Hospital's Rule is used

$$\overset{L}{=} \lim_{x \to 2} \frac{2x+1}{2x}$$ now let x go to 2

$$= \frac{2(2)+1}{2(2)} = \boxed{\frac{5}{4}}$$ same as the "old way"

example

$$\lim_{x\to 0} \frac{x+\sin 2x}{x-\sin 2x} \to \frac{0}{0}$$

MUST see $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in order to use l'Hospital's Rule

$$\stackrel{L}{=} \lim_{x\to 0} \frac{1+(\cos 2x)(2)}{1-(\cos 2x)(2)}$$

as $x \to 0$, $1+2\cos 2x = 3$

$1-2\cos 2x = -1$

so, the limit is not going to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ anymore

$$= \frac{3}{-1} = \boxed{-3}$$

example

$$\lim_{x\to 0} \frac{x+\sin 2x}{x-\cos 2x} \to \frac{0}{-1}$$

NOT $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so do NOT use l'Hospital's Rule

$$= \frac{0}{-1} = \boxed{0}$$

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<u>example</u> which of the following grows faster as $x \to \infty$

$f(x) = x^2$ or $g(x) = 2^x$

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^2}{2^x} = 0$ then that means 2^x is much bigger than x^2 as $x \to \infty$, so 2^x grows faster

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^2}{2^x} = \infty$ then x^2 is much bigger than 2^x as $x \to \infty$, so x^2 grows faster

$\lim\limits_{x\to\infty} \frac{x^2}{2^x} \to \frac{\infty}{\infty} = ?$ But since we see $\frac{\infty}{\infty}$, we can use l'Hospital's Rule

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$$\lim_{x\to\infty} \frac{x^2}{2^x}$$

$$\stackrel{L}{=} \lim_{x\to\infty} \frac{2x}{2^x \cdot \ln 2} \rightarrow \frac{\infty}{\infty}$$

$$\frac{d}{dx}(2^x) = 2^x \cdot \ln 2$$

we see $\frac{\infty}{\infty}$ again, so we use l'Hospital's Rule again (keep using it until we no longer see $\frac{\infty}{\infty}$ or $\frac{0}{0}$)

$$\stackrel{L}{=} \lim_{x\to\infty} \frac{2}{(2^x \cdot \ln 2)\cdot \ln 2} \rightarrow \frac{2}{\infty}$$

DO NOT use l'Hospital the third time

$$= \boxed{0}$$

So, 2^x grows faster than x^2

Other indeterminate forms: $\underbrace{\infty - \infty}_{\text{which } \infty \text{ is bigger?}}$, $\underbrace{1}_{(\#\text{ close to } 1)}{}^{\infty \,(\text{big } \#)}$, $\underbrace{\infty \cdot 0}_{(\text{big }\#)(\text{small }\#)}$

<u>example</u>

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) \to \infty - \infty = ?$$

transform this into $\frac{\infty}{\infty}$ or $\frac{0}{0}$ and then use l'Hospital's Rule

$$= \lim_{x \to 0^+} \left(\frac{e^x - 1}{x(e^x - 1)} - \frac{x}{x(e^x - 1)} \right)$$

$$= \lim_{x \to 0^+} \frac{e^x - 1 - x}{x(e^x - 1)} \to \frac{0}{0}$$

we can now use l'Hospital's Rule

$$\overset{L}{=} \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + (e^x - 1)}$$

product rule in denominator

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$$= \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + (e^x - 1)} \to \frac{0}{0}$$ l'Hospital's again

$$\overset{L}{=} \lim_{x \to 0^+} \frac{e^x}{xe^x + e^x + e^x} \to \frac{1}{2}$$ Stop, no more l'Hospital's Rule

$$= \boxed{\frac{1}{2}}$$

<u>example</u> $\lim\limits_{x\to 0^+} (1+x)^{\cot x} \to 1^{\infty}$

transform into $\frac{\infty}{\infty}$ or $\frac{0}{0}$ then use l'Hospital's Rule

$\lim\limits_{x\to 0^+} \underbrace{(1+x)^{\cot x}}_{y}$ so we want $\lim\limits_{x\to 0^+} y$

$$\ln y = \ln (1+x)^{\cot x}$$

$$= \cot x \cdot \ln(1+x) = \frac{\ln(1+x)}{\tan x}$$

now note $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \frac{\ln(1+x)}{\tan x} \to \frac{0}{0}$ l'Hospital's is applicable here

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$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{\tan x} \to \frac{0}{0}$$

$$\stackrel{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{\sec^2 x} \to \frac{1}{1}$$

Stop, no more l'Hospital's Rule

$$= 1$$

we are not done! we found $\lim_{x\to 0^+} \ln y$ but we want $\lim_{x\to 0^+} y$

$$\lim_{x\to 0^+} \ln y = 1 \quad \text{and} \quad e^{\ln y} = y$$

$$\lim_{x\to 0^+} e^{\ln y} = e^1 \to \lim_{x\to 0^+} y = \boxed{e}$$

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<u>example</u> $\lim_{x\to\infty} x^2\left(\frac{1}{x} - \sin\frac{1}{x}\right) \to \infty \cdot 0$

transform into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then use l'Hospital's Rule

$$\lim_{x\to\infty} x^2\left(\frac{1}{x} - \sin\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x^2}} \to \frac{0}{0}$$

$$a \cdot b = \frac{b}{\frac{1}{a}} = \frac{b \cdot a}{\frac{1}{a} \cdot a} = \frac{ab}{1} = ab$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{-\frac{1}{x^2} - \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}$$

$$\frac{1}{x^2} = x^{-2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = -2x^{-3}$$

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$$= \lim_{x\to\infty} \frac{-\frac{1}{x^2} - \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \cdot \frac{-x^2}{-x^2}$$

$$= \lim_{x\to\infty} \frac{1-\cos\left(\frac{1}{x}\right)}{\frac{2}{x}} \to \frac{0}{0}$$

l'Hospital's Rule again

$$\overset{L}{=} \lim_{x\to\infty} \frac{- -\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^2}} = \lim_{x\to\infty} \frac{\left(-\frac{1}{x^2}\right)\left(\sin\frac{1}{x}\right)}{-\frac{2}{x^2}} \cdot \frac{-x^2}{-x^2}$$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{2} \to \frac{0}{2}$$

Stop, no more l'Hospital's Rule

$$= \boxed{0}$$

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