Draw Erase Select Point Add Undo Redo Rec Stop View Close

4.9 Antiderivatives

antiderivative — is the reverse of derivative

$\rightarrow$ given $f(x)$ find $F(x)$ such that $F'(x) = f(x)$

for example, $F(x) = -\cos x$ is antiderivative of $f(x) = \sin x$

because $F' = -(-\sin x) = \sin x = f(x)$

Antiderivative of x^n

an antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ if $n \neq -1$

why $\dfrac{x^{n+1}}{n+1}$? because $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = \dfrac{1}{n+1}\dfrac{d}{dx}(x^{n+1})$

$$= \frac{1}{n+1}(n+1)\cdot x^n = x^n$$

for example, an antiderivative of x^2 is $\dfrac{x^{2+1}}{2+1} = \dfrac{x^3}{3} = \dfrac{1}{3}x^3$

Check: $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3}\cdot 3x^2 = x^2$

but notice $\frac{1}{3}x^3 + 1$ is <u>also</u> an antiderivative of x^2

because $\frac{d}{dx}\left(\frac{1}{3}x^3+1\right) = \frac{1}{3}\cdot 3x^2 + 0 = x^2$

in fact, so is $\frac{1}{3}x^3+5,\ \frac{1}{3}x^3-10,\ \frac{1}{3}x^3+\pi,\ \frac{1}{3}x^3+e^{\pi}+55$

all have derivatives equal to x^2

all of these are possible antiderivatives of x^2

all of these are in form of $\frac{1}{3}x^3 + C$

C some constant

$$\boxed{\text{so, the most general antiderivative of } x^n \ (n \neq -1) \text{ is } \frac{x^{n+1}}{n+1} + C}$$

example Find the antiderivative of $f(x) = -\frac{10}{x^{12}}$

what function $F(x)$ gave us this as its derivative?

rewrite: $f(x) = -\frac{10}{x^{12}} = -10\,\underbrace{x^{-12}}_{x^n \text{ part}}$

constant multiple
doesn't participate in the process
(just like differentiation)

$$F(x) = -10\left(\frac{x^{-12+1}}{-12+1}\right) + C$$

"constant of integration"

$$= \frac{-10}{-11}x^{-11} + C$$

$$= \boxed{\frac{10}{11}x^{-11} + C}$$

check: is $\frac{d}{dx}\left(\frac{10}{11}x^{-11} + c\right) = -\frac{10}{x^{12}}$?

$$\frac{10}{11}\cdot -11x^{-12} = -10x^{-12}$$

so, yes

the <u>indefinite integral</u> is used to denote the process of finding the antiderivative

$$\int f(x)\,\underline{\underline{dx}} = F(x) + C$$

integral sign

part of the notation (dx and $\int$ sandwich the thing whose antiderivative you want)

so, from last example, we know $\int -\frac{10}{x^{12}}\,dx = \frac{10}{11}x^{-11} + C$

since finding the antiderivative is closely related to differentiation, many rules carry over to this

for example, we can handle things separated by + or − separately

Draw Erase Select Point Add Undo Redo Rec Stop View Close

<u>example</u> $$\underbrace{\int \left(\frac{2}{\sqrt{x}} + 2\sqrt{x} \right) dx}$$

what function has this as derivative?

rewrite:

$$\int \left(2x^{-1/2} + 2x^{1/2} \right) dx$$

$$= 2\left(\frac{x^{-1/2+1}}{-1/2+1} \right) + 2\left(\frac{x^{1/2+1}}{1/2+1} \right) + C$$

just need one $+C$

$$= 2\left(\frac{x^{1/2}}{1/2} \right) + 2\left(\frac{x^{3/2}}{3/2} \right) + C$$

$$= 4x^{1/2} + 2 \cdot \frac{2}{3} x^{3/2} + C = \boxed{4x^{1/2} + \frac{4}{3} x^{3/2} + C}$$

$2 \cdot \frac{2}{1}$

example

$$\int \left[\frac{x^7+x^5}{x^4} + (x+1)^2\right] dx$$

rewrite as combinations of x^n

$$= \int \left[\frac{x^7}{x^4} + \frac{x^5}{x^4} + (x^2+2x+1)\right] dx$$

$$= \int (x^3 + x + x^2 + 2x + 1)\, dx = \int (x^3 + x^2 + 3x^1 + \overset{1\cdot x^0}{1})\, dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + 3\cdot\frac{x^2}{2} + 1\cdot\frac{x^1}{1} + C$$

$$= \boxed{\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + C}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1$$

but what happens with $n = -1$?

$$\int x^{-1} \, dx = \int \underbrace{\frac{1}{x}} \, dx$$

what has
this as derivative?
$\rightarrow \ln x$

So, $\boxed{\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C}$

absolute value to make
$\ln|x|$ defined.

Draw Erase Select Point Add Undo Redo Rec Stop View Close

with trig functions, need to think of the differentiations rules in reverse

$$\int \cos x \, dx = \sin x + C \quad \text{because} \quad \frac{d}{dx}(\sin x + C) = \cos x$$

$$\int \sin x \, dx = -\cos x + C \quad \text{because} \quad \frac{d}{dx}(-\cos x + C) = -(-\sin x) = \sin x$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C \quad \text{because} \quad \frac{d}{dx}(-\csc x + C)$$

$$= -\frac{d}{dx}(\csc x)$$

$$= -(-\csc x \cot x)$$

$$= \csc x \cot x$$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

<u>example</u>

$$\int (\sin x + \sec x \tan x)\, dx$$

separated by + or −, so we can deal w/ them separately

$$= \int \sin x\, dx + \int \sec x \tan x\, dx$$

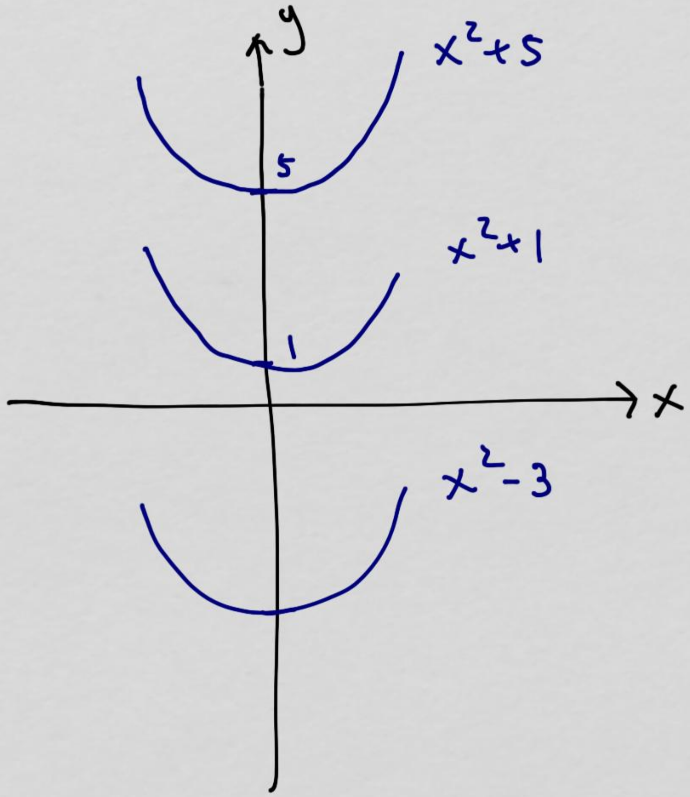
$$= \boxed{-\cos x + \sec x + C}$$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

to find what the actual value of C is, we need to know one point that the antiderivative passes through

for example, $\int 2x\,dx = 2\cdot\left(\frac{x^2}{2}\right)+C = \underbrace{x^2+C}$

family of parabolas w/ vertex at $y=C$



all of these are candiates, but which one is it?

if we know one point the "real one" passes through, then we know

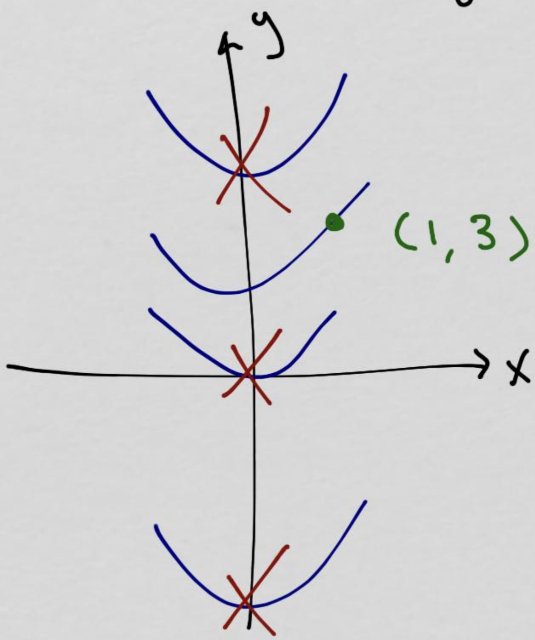
example

$f(x) = 2x$

find $F(x)$ such that $F'(x) = f(x)$ and $F(1) = 3$

one point on the true antideriv.

$$F(x) = \int 2x \, dx = 2 \cdot \frac{x^2}{2} + C = x^2 + C$$



$$F(x) = x^2 + C$$

$$F(\underset{\uparrow\, x}{1}) = \underset{\uparrow\, y}{3}$$

$$3 = (1)^2 + C$$

$$3 = 1 + C$$

$C = 2$, so, $\boxed{F(x) = x^2 + 2}$

we handle higher-order derivatives the same way, but there are more constants and more points are needed to find those constants

<u>example</u> $F''(x) = x$ find $F(x)$

$$F'(x) = \int x \, dx$$ this undoes one derivative

$$F'(x) = \frac{x^2}{2} + C$$

now undo one more derivative

$$F(x) = \int \left(\frac{x^2}{2} + C\right) dx = \int \left(\tfrac{1}{2}x^2 + Cx^0\right) dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right) + C\left(\frac{x^1}{1}\right) + D$$ the 2nd constant of integration (don't reuse letters)

$$= \boxed{\frac{1}{6}x^3 + Cx + D}$$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

to find C and D, we need to know one point on $F(x)$

and one point on $F'(x)$

<u>example</u> $F''(x) = \sin x$ and $\boxed{F'(0)=3}$, $\boxed{F(0)=4}$

$$F'(x) = \int \sin x \, dx = -\cos x + C$$

$$F'(x) = -\cos x + C$$

$$3 = -\cos(0) + C = -1 + C \rightarrow \quad C = 4$$

update: $F'(x) = -\cos x + 4$

$$F(x) = \int (-\cos x + 4)\,dx = -\sin x + 4x + D$$

$$4 = -\sin(0) + 4(0) + D \rightarrow D = 4$$

update: $\boxed{F(x) = -\sin x + 4x + 4}$