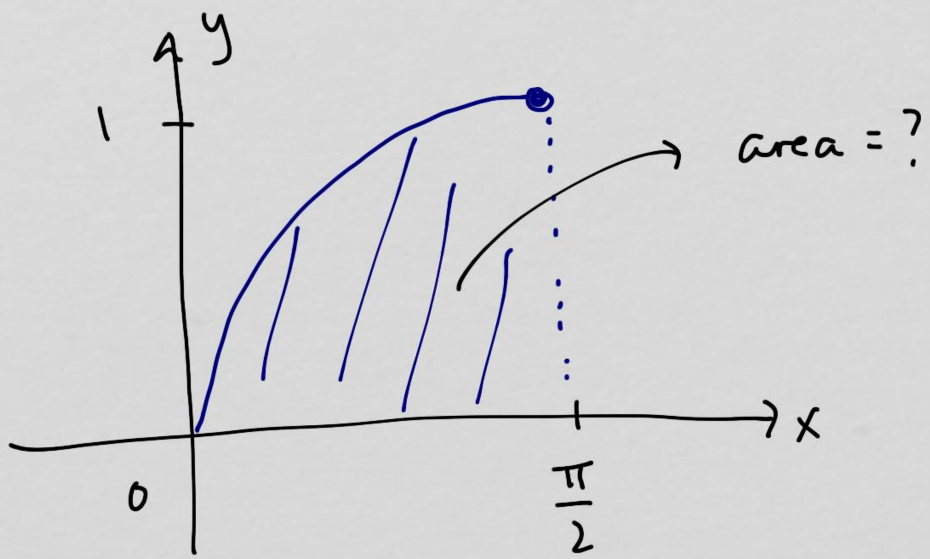
5.1 Approximating Areas Under Curves

How to find the area under $f(x)$ $(f(x) \geq 0)$ on $a \leq x \leq b$

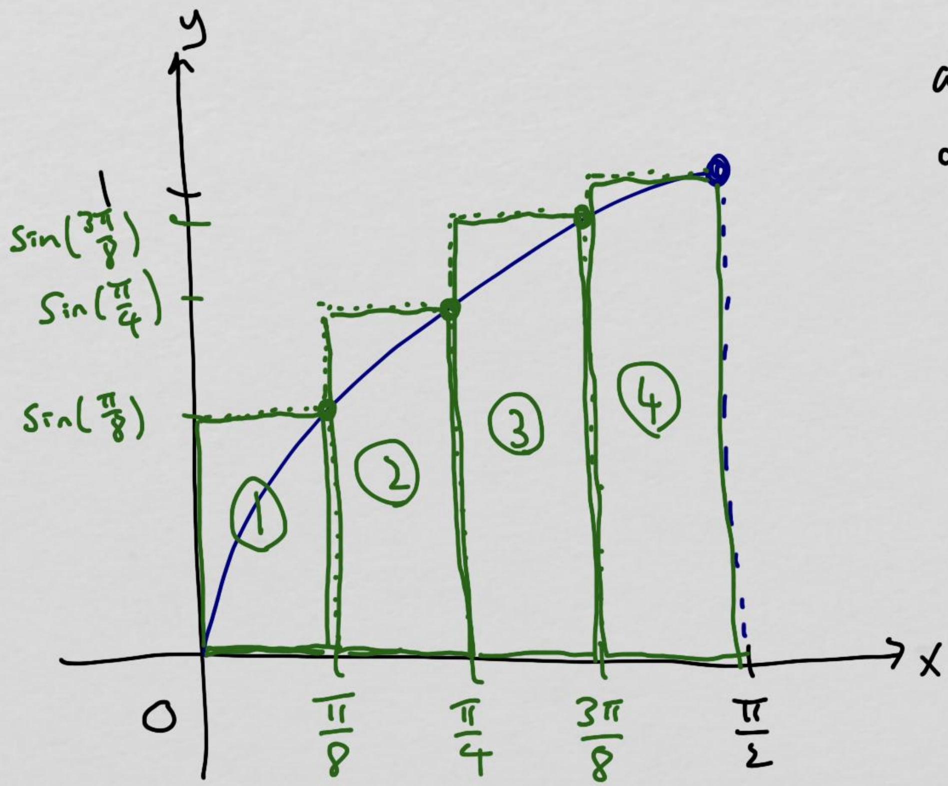
for example, what is the area under the curve $y = \sin x$ on $[0, \frac{\pi}{2}]$?



one way to approximate the area is to use a <u>Riemann Sum</u>

- divide the region into a bunch of rectangles, then sum the areas of the rectangles

$f(x) = \sin x$ on $\left[0, \frac{\pi}{2}\right]$



First, we decide how many rectangles to use

as an example, let's use $\boxed{4}$

divide the interval $\left[0, \frac{\pi}{2}\right]$ into 4 equal parts

grid points: $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$

Then we decide where on each subinterval to sample the rectangle heights

typical choices: Right endpoint
Left endpoint
Midpoint

here, as an example, let's use Right endpoints

right end

4 sub-intervals

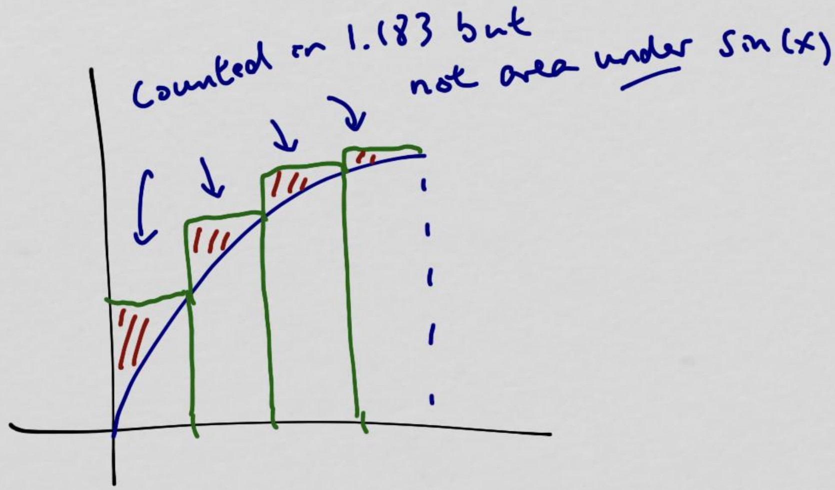
$$R_4 = \underbrace{\left(\frac{\pi}{8}\right)}_{\text{width of } 1} \underbrace{\sin\left(\frac{\pi}{8}\right)}_{\text{height of } 1} + \underbrace{\left(\frac{\pi}{8}\right)}_{\text{width of } 2} \underbrace{\sin\left(\frac{\pi}{4}\right)}_{\text{height of } 2} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)}_{3} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{2}\right)}_{4} = \boxed{1.183}$$

so, we estimate the area under $\sin(x)$ on $[0, \frac{\pi}{2}]$

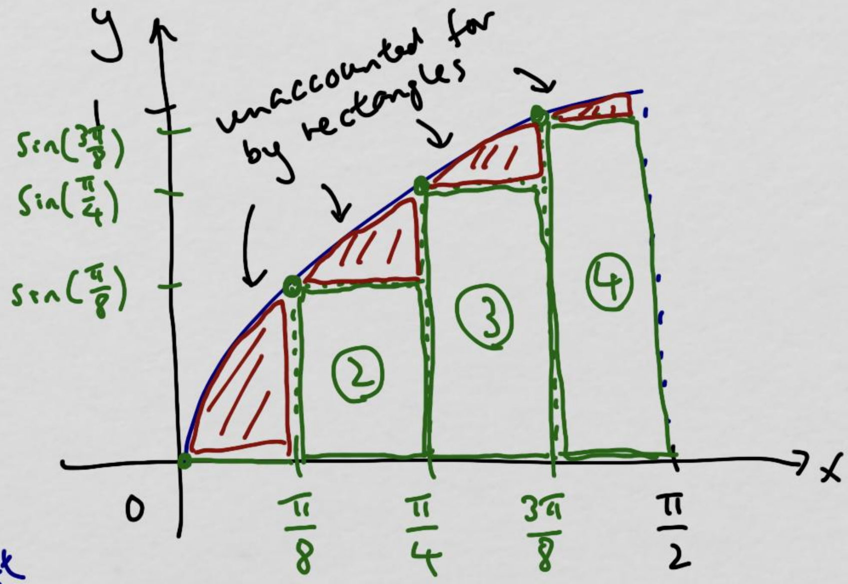
to be around 1.183

we know this value is an <u>overestimate</u> because each

rectangle has part of it above the curve $\sin(x)$



now let's try the same thing w/ Left endpoints, keep using $\boxed{4}$ rectangles



note the left end of the first subinterval is 0, so height of rectangle ① is $\sin(0) = 0$

Left

$$L_4 = \underbrace{\left(\frac{\pi}{8}\right)\sin(0)}_{1} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)}_{2} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{4}\right)}_{3} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)}_{4} = \boxed{0.791}$$

4 rectangles

the estimate of 0.791 is an underestimate because there are unaccounted for regions under $\sin(x)$ where the rectangles don't reach.

we now can confidently say, whatever the true area under $\sin(x)$ on $[0, \frac{\pi}{2}]$ is, it is no smaller than 0.791 and no bigger than 1.183

$$0.791 \leq \text{true area} \leq 1.183$$

now let's try the same problem again, but use the midpoint of each subinterval to find the heights

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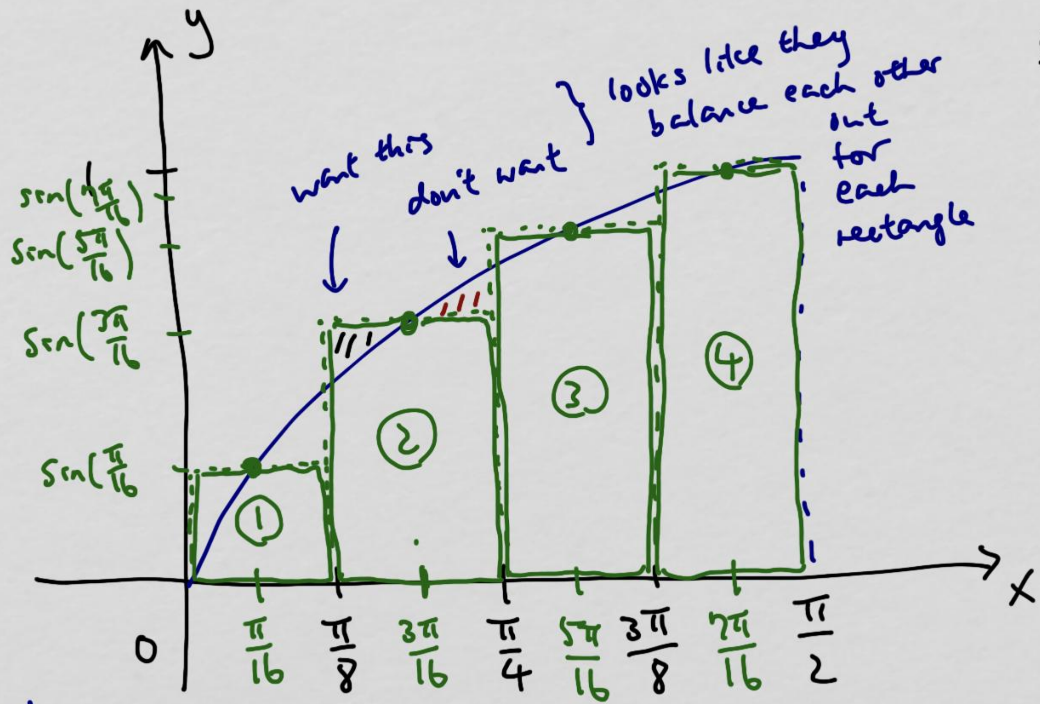


Still use $\boxed{4}$ rectangles

but use midpoints

on first subinterval, midpoint is $x = \frac{\pi}{16}$

on second subinterval, midpoint is $x = \frac{3\pi}{16}$

on 3rd, midpoint is $x = \frac{5\pi}{16}$

on 4th, midpoint is $x = \frac{7\pi}{16}$

note each rectangle still has width of $\frac{\pi}{8}$

mid

4 rectangles

$$M_4 = \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{16}\right)}_{1} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{3\pi}{16}\right)}_{2}$$

$$+ \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{5\pi}{16}\right)}_{3} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{7\pi}{16}\right)}_{4} = 1.006$$

So, the midpoints seem to give us a value somewhere between the left endpoints and the right endpoint, probably closest to the true area.

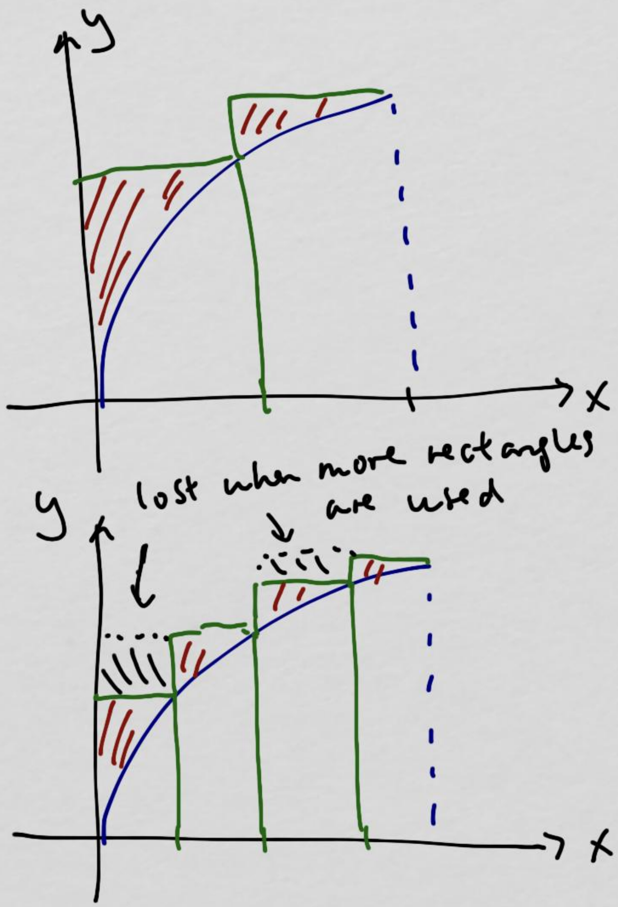
as a reference, the true area under $y = \sin(x)$ on $[0, \frac{\pi}{2}]$ is 1 (we will learn how to do this later in ch. 5)

if $f(x)$ is an increasing function on (a, b), then using the Right endpoints will result in an overestimate

if $f(x)$ is an increasing function on (a, b), then using the Left endpoints will result in an underestimate

(the opposite is true if $f(x)$ is decreasing)

the estimate, no matter which endpoint or midpoint, gets better the more rectangles are used



excess areas are smaller

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In general, regardless of the choice of left/right/mid points, we can write the approximation as

$$\Delta x = \frac{b-a}{n}$$

right end of interval

left end of interval

\# of rectangles

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

sample points (left/right/mid point of each subinterval)

upper limit

$$= \sum_{k=1}^{n} f(x_k)\Delta x$$

lower limit

Summation (sigma) notation means to add up all terms starting at $\underline{k=1}$ ending at $\underline{k=n}$

for example,

$$\sum_{k=1}^{5} k^2 = \underbrace{(1)^2}_{k=1} + \underbrace{(2)^2}_{k=2} + \underbrace{(3)^2}_{k=3} + \underbrace{(4)^2}_{k=4} + \underbrace{(5)^2}_{k=5}$$

$$= 1+4+9+16+25 = 55$$

matches the # at top of Σ

STOP

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$ can be written as

$k=1$ $k=2$ $k=3$ $k=4$ $k=5$

$$= \sum_{k=1}^{5} \frac{1}{k+2}$$

note each denominator is 2 more than k