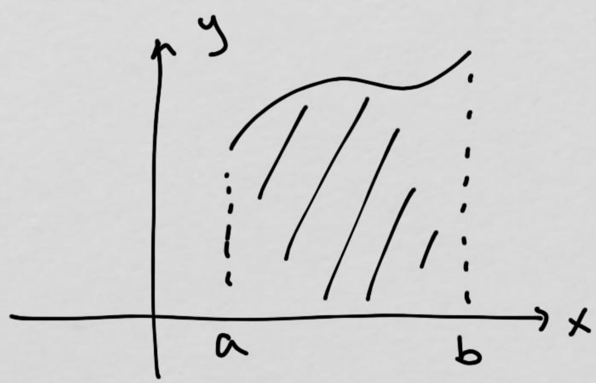
5.2 Definite Integrals

last time: if $f(x) \ge 0$, then the area underneath it from $x=a$ to $x=b$



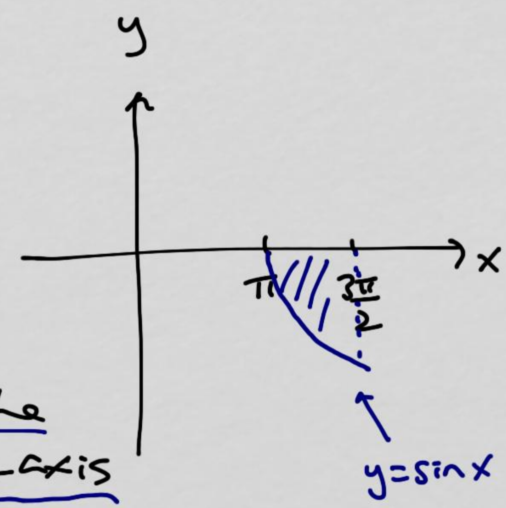
can be approximated by Riemann sum.

what if $f(x) < 0$?

for example, $f(x) = \sin x$ on $[\pi, \frac{3\pi}{2}]$

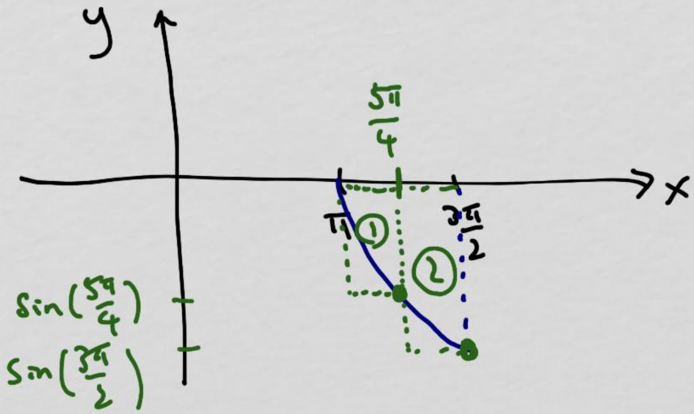
we can no longer talk about the area under the curve (it goes to $-\infty$)

but we can still find the area between the curve and the x-axis



example $f(x) = \sin x$ on $\left[\pi, \frac{3\pi}{2}\right]$

right end points, $n=2$ rectangles



$$R_2 = \underbrace{\left(\frac{\pi}{4}\right)}_{\text{width of } 1} \underbrace{\sin\left(\frac{5\pi}{4}\right)}_{\text{height of } 1} + \underbrace{\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{2}\right)}_{2}$$

$$= \left(\frac{\pi}{4}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\pi}{4}\right)(-1) = \boxed{-1.341}$$

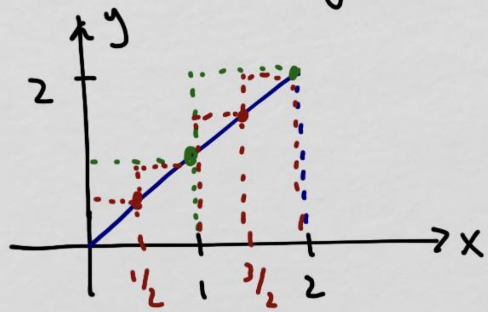
area between $\sin x$ and x-axis on $\left(\pi, \frac{3\pi}{2}\right)$

note the "heights" are negative

note if $f(x) < 0$ (region is below x-axis), the area is negative.

Approximation of the area improves as more rectangles are used.

<u>example</u> Area under $y = x$ on $[0, 2]$



it's a triangles, so we can use geometry to find the <u>exact</u> area

$$A = \frac{1}{2}(2)(2) = 2$$

as more rectangles are used, we should see approximations approach the exact value.

green $R_2 = (1)(1) + (1)(2) = 3$

red $R_4 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{2}\right)(2) = 2.5$

$$R_8 = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)(1)$$
$$+ \left(\frac{1}{4}\right)\left(\frac{5}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{7}{4}\right) + \left(\frac{1}{2}\right)(2) = 2.25$$

so, as n increases, $R_n \to$ true area (same for L_n, M_n)

\# of rectangles — left end — midpoint

furthermore, as $n \to \infty$, R_n, L_n, M_n all eventually go to the true value

sample points (L/R/M)

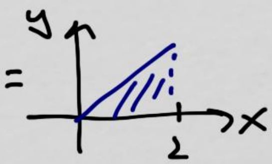
or $$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k)\,\Delta_k = \text{true area}$$

width of each rectangle $\left(\frac{b-a}{n}\right)$

that cumbersome expression can be neatly expressed as

the <u>Definite Integral</u>

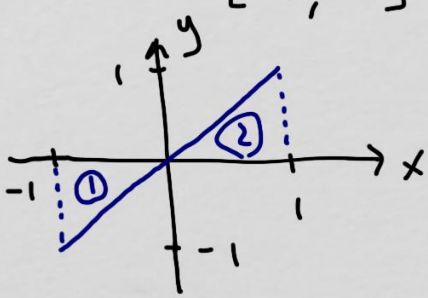
$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

this represents the area between f(x) and the x-axis on [a,b]. For example, $\int_0^2 x\,dx =$  $= 2$ (exactly)

<u>example</u>

$$\int_{-1}^{1} x \, dx$$

this represents the area between $f(x) = x$ and the x-axis on $[-1, 1]$



note $\int_{-1}^{1} x \, dx = \underbrace{\text{"area" of } ①}_{\substack{\text{under } x\text{-axis} \\ \text{(negative area)}}} + \text{area of } ②$

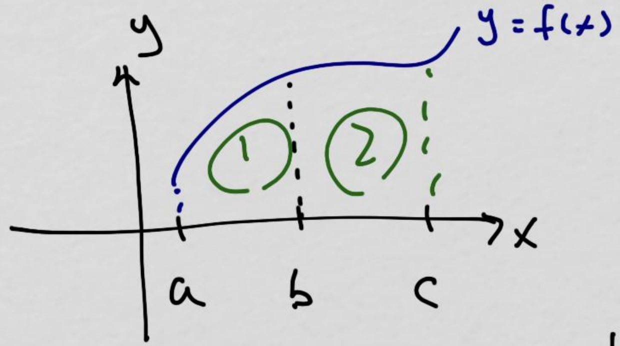
$= -\frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 0$

so, we know $\int_{-1}^{1} x \, dx = 0$

also, note $\underbrace{\int_{-1}^{0} x \, dx}_{①} + \underbrace{\int_{0}^{1} x \, dx}_{②} = \int_{-1}^{1} x \, dx$

this gives us a useful property : $\underbrace{\int_a^b f(x)\,dx}_{1} + \underbrace{\int_b^c f(x)\,dx}_{2} = \underbrace{\int_a^c f(x)\,dx}_{\text{the whole thing}}$

for a c such that $a < b < c$



other useful properties :

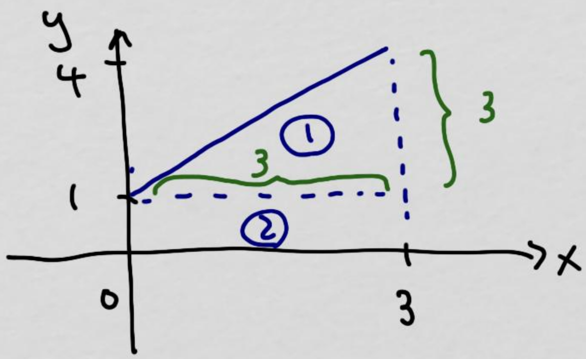
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

why? Let's use another simple example and illustration to see why it is true

<u>example</u>

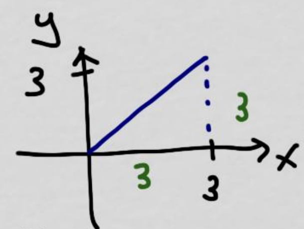
$$\int_0^3 (x+1)\,dx$$



notice it can be divided into a triangle and a rectangle

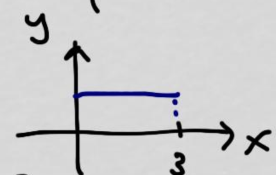
1 height 3 base 3

2



the triangle has the same area as $\int_0^3 x\,dx$

the rectangle has the same area as $\int_0^3 1\,dx$



so, this shows $$\int_0^3 (x+1)dx = \int_0^3 x\,dx + \int_0^3 1\,dx$$

other sometimes useful properties:

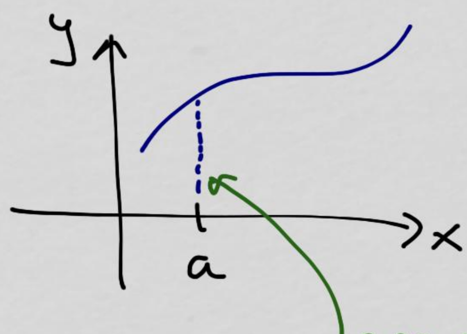
$$\int_a^b C \cdot f(x)\,dx = C \int_a^b f(x)\,dx$$

some constant

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

we can switch the places of upper and lower integration limits by changing the sign of the integral

$$\int_a^a f(x)\,dx = 0$$



area of this line is

$$\int_a^a f(x)dx = 0$$