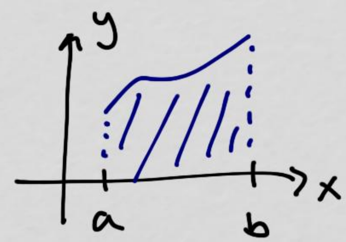
5.3 The Fundamental Theorem of Calculus

last time: $\int_a^b f(x)\,dx$ represents the area between $f(x)$ and x-axis

from $x = a$ to $x = b$

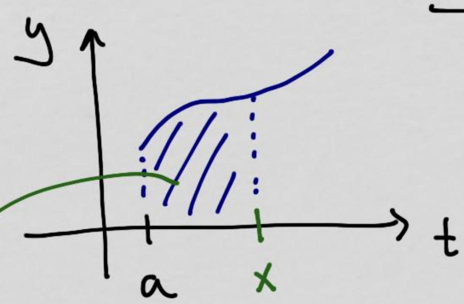


now let's consider an area function

$$g(x) = \int_a^x f(t)\,dt$$

"dummy variable"

(we don't want to use x to mean two different things)



here, x is the right end of the interval which we can move

(we don't want to use x again for the function's domain)

$$\int_a^x f(t)\,dt = g(x)$$

$$g(x) = \int_a^x f(t)\,dt$$

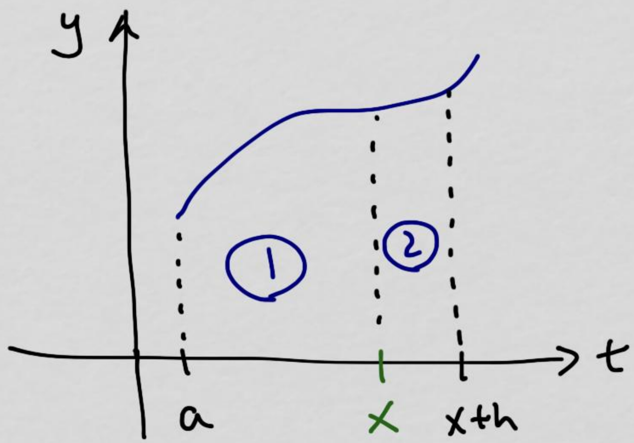
what is $g'(x)$?

what is the rate of change of the accumulated area as x changes?

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

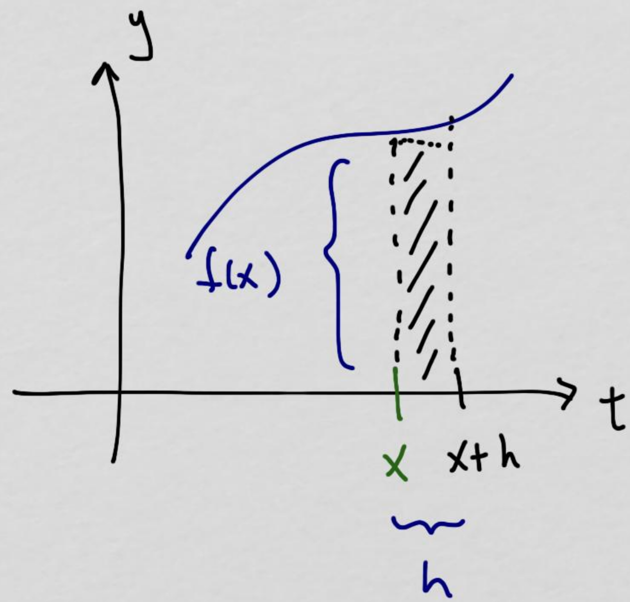
$$g(x) = \int_a^x f(t)\,dt \quad ①$$

$$g(x+h) = \int_a^{x+h} f(t)\,dt \quad ① + ②$$



so, $$g(x+h) - g(x) = \underbrace{\int_a^{x+h} f(t)\,dt}_{① + ②} - \underbrace{\int_a^{x} f(t)\,dt}_{①} = ②$$

when h is very small $\left(\lim_{h\to 0}\right)$, ② is approximately a rectange



So, (2) is roughly a rectangle w/ height $f(x)$ and width h

So, $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\overbrace{f(x)\cdot \cancel{h}}^{(2)}}{\cancel{h}} = f(x)$

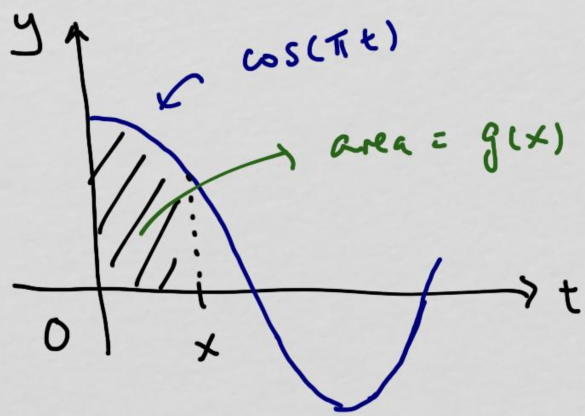
$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

This is called the <u>Fundamental Theorem of Calculus</u> (part 1)

(FTC 1)

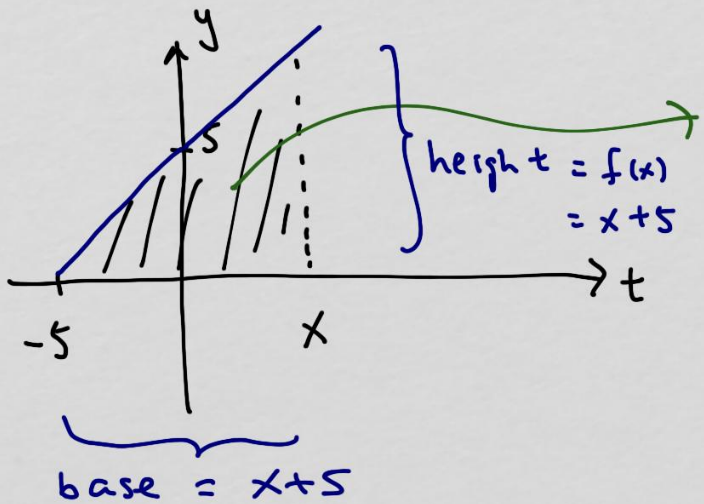
<u>Example</u>

$$g(x) = \int_0^x \cos(\pi t)\,dt$$



$$g'(x) = \frac{d}{dx}\int_0^x \cos(\pi t)\,dt = \boxed{\cos(\pi x)}$$

example Find the rate of change of the function representing the area under $f(t) = t + 5$ from $t = -5$ to $t = x$



$$\text{area} = A(x) = \int_{-5}^{x} (t+5)\,dt$$

by geometry, $A(x) = \frac{1}{2}(x+5)(x+5)$

$$= \frac{1}{2}(x+5)^2$$

$$A'(x) = \frac{d}{dx}\int_{-5}^{x} \underline{(t+5)}\,dt = \underline{x+5}$$

in this case, we can actually verify A' w/ our A from geometry: $A = \frac{1}{2}(x+5)^2 \rightarrow A' = \frac{1}{2}\cdot 2(x+5) = x+5$

(so FTC 1 does give us what is correct)

Draw Erase Select Point Add Undo Redo Rec Stop View Close

FTC 1: $\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)$ only works if the variable (x) is <u>the upper limit of integration</u>

if it's not, swap w/ the other one by using the property

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

<u>example</u> $\frac{d}{dx}\int_{x}^{1}\sqrt{t^5+1}\,dt$ x is not the upper limit

$$= \frac{d}{dx}\left(-\int_1^x \sqrt{t^5+1}\,dt\right)$$

$$= -\underbrace{\frac{d}{dx}\int_1^x \sqrt{t^5+1}\,dt}_{\text{apply FTC 1}} = \boxed{-\sqrt{x^5+1}}$$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

what if that x is not just x but some function of x ?

example

$$y = \int_2^{e^{3x}} \sin^2(5t)\,dt \qquad y' = ?$$

So, x is at the upper limit position, no need to swap

but it's not just x

so, to use FTC 1, we need Chain Rule

let $u = e^{3x}$

$$y = \int_2^{u} \sin^2(5t)\,dt = y(u) \quad \text{with } u = e^{3x}$$

by Chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

$$y = \int_2^u \sin^2(5t)\,dt \qquad u = e^{3x}$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \underbrace{\left(\frac{d}{du}\int_2^{u} \underline{\sin^2(5t)}\,dt\right)}_{\text{apply FTC 1 directly}} \left(\frac{d}{dx}e^{3x}\right)$$

change to u

$$= \sin^2(5u)\cdot 3e^{3x}$$

Sub u out

$$= \sin^2(5e^{3x})\cdot 3e^{3x}$$

$$= \boxed{3e^{3x}\sin^2(5e^{3x})}$$

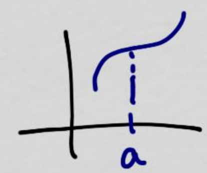
The second part of Fundamental Theorem of Calculus allows us to calculate $\int_a^b f(x)\,dx$ exacty (no more Riemann Sum)

FTC 1: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

so, the antiderivative of $\frac{d}{dx}\int_a^x f(t)\,dt$ is the antiderivative of $f(x)$

$$\rightarrow \int_a^x f(t)\,dt = F(x) + C \quad \text{where } F'(x) = f(x)$$

we know, from last time, $\int_a^a f(t)\,dt = 0$



so, $\int_a^a f(t)\,dt = F(a) + C = 0 \longrightarrow \boxed{C = -F(a)}$

then, from $\displaystyle\int_a^x f(t)\,dt = F(x) + C$

we get $\displaystyle\int_a^b f(t)\,dt = F(b) + C$ but we know $C = -F(a)$

$$= F(b) - F(a)$$

therefore, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

where $F' = f$

Fundamental Theorem of Calculus (part 2)

(FTC 2)

<u>example</u>

$$\int_0^3 \underbrace{2x}_{f(x)}\,dx$$

FTC 2: $\int_a^b f(x)\,dx = F(b) - F(a)$

find $F(x)$ such that $F'(x) = f(x) = 2x$

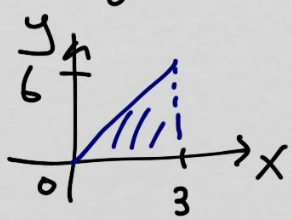
$$F(x) = 2 \cdot \frac{x^2}{2} + C = x^2 + C$$

by FTC2, this is equal to $F(b) - F(a)$, with $b = 3$ and $a = 0$

$$= (3^2 + C) - (0^2 + C)$$

$$= 9 + \underbrace{C - C}_{\text{note } C \text{ disappears}} = \boxed{9}$$

check: $\int_0^3 2x\,dx$



area $= \frac{1}{2}(3)(6) = \frac{18}{2} = 9$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

notation: sometimes FTC 2 is written as

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

example $\displaystyle\int_{-1}^{2} \underbrace{(x^2-2x+7)}_{f(x)}\,dx$

since $+C$ disappears at the end, we no longer need to include it

$$= F(x)\Big|_{-1}^{2} = \frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 7x\,\Big|_{-1}^{2}$$

$$= \left[\frac{2^3}{3} - 2^2 + 7(2)\right] - \left[\frac{(-1)^3}{3} - (-1)^2 + 7(-1)\right]$$

$$= \left(\frac{8}{3} - 4 + 14\right) - \left(-\frac{1}{3} - 7 - 7\right)$$

$$= \boxed{21}$$