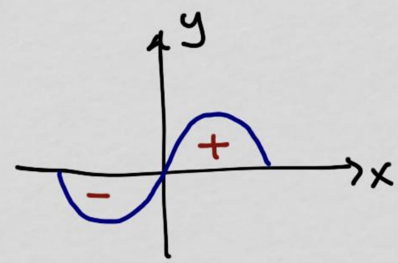
5.4 Working with Integrals

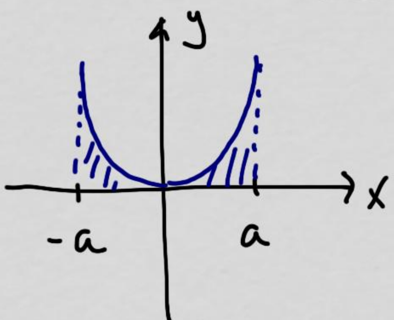
we know $\int_a^b f(x)\,dx$ gives us the area between $f(x)$ and x-axis on $[a, b]$

if $f(x) \geq 0$, then area is positive

if $f(x) < 0$, then area is negative

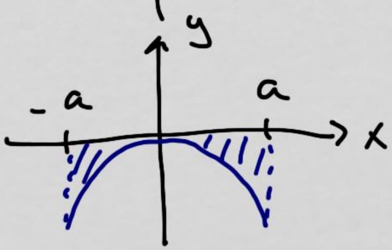


If $f(x)$ is <u>even</u> : $f(-x) = f(x) \rightarrow$ <u>y-axis symmetry</u>



notice $\int_{-a}^{0} f(x)\,dx = \int_0^a f(x)\,dx$

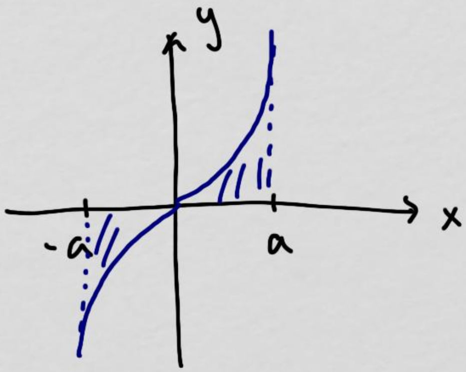
and since $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

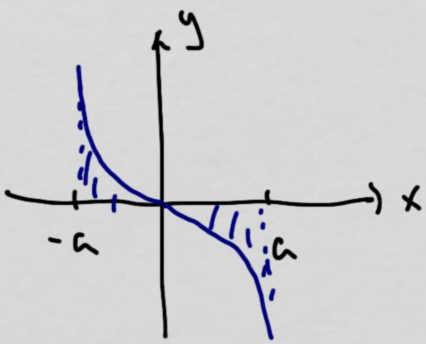


we see that $\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{0} f(x)\,dx = 2\int_0^a f(x)\,dx}$

if $f(x)$ is even

if $f(x)$ is odd : $f(-x) = -f(x) \rightarrow$ origin symmetry





notice $\displaystyle\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$

so, $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$\displaystyle = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

if $f(x)$ is odd

<u>example</u> $\int_{-1}^{1} (x^4+3)\,dx$

$f(x) = x^4+3$ is <u>even</u> because $f(-x) = (-x)^4+3 = x^4+3 = f(x)$

so $\int_{-1}^{1}(x^4+3)\,dx = 2\int_{0}^{1}(x^4+3)\,dx$ $\left(\text{or } 2\int_{-1}^{0}(x^4+3)\,dx\right)$

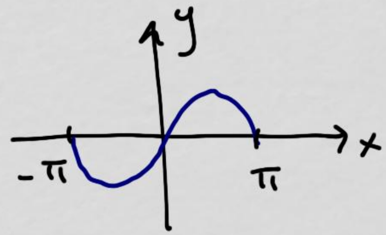
$$= 2\left(\frac{x^5}{5}+3x\right)\Big|_0^1 = 2\left[\left(\frac{1^5}{5}+3\cdot 1\right) - \left(\frac{0^5}{5}+3\cdot 0\right)\right]$$

$$= 2\left(\frac{1}{5}+3\right) = 2\cdot\frac{16}{5} = \boxed{\frac{32}{5}}$$

check: $\int_{-1}^{1}(x^4+3)\,dx = \left(\frac{x^5}{5}+3x\right)\Big|_{-1}^{1}$

$$= \left(\frac{1^5}{5}+3\cdot 1\right) - \left(\frac{(-1)^5}{5}+3\cdot -1\right) = \frac{16}{5} - \left(-\frac{16}{5}\right) = \frac{32}{5}$$

<u>example</u> $\int_{-\pi}^{\pi} \sin x \, dx$



$\sin x$ is odd, so $\displaystyle\int_{-\pi}^{\pi} \sin x \, dx = \boxed{0}$

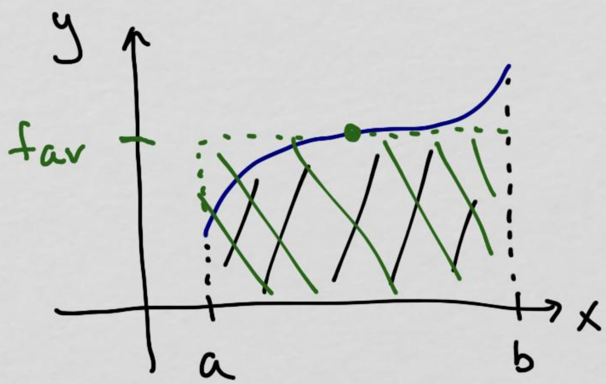
$\left(\text{because } \displaystyle\int_{-\pi}^{0} \sin x \, dx = -\int_{0}^{\pi} \sin x \, dx\right)$

check: $\displaystyle\int_{-\pi}^{\pi} \sin x \, dx = (-\cos x)\Big|_{-\pi}^{\pi}$

$= -\cos(\pi) - -\cos(-\pi)$

$= -(-1) + (-1) = 1 - 1 = 0$

Definite integral can be used to find the average value of $f(x)$ on $[a, b]$



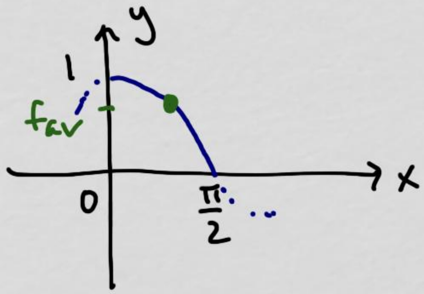
black shaded region : $\int_a^b f(x)\,dx$

f_{av} : average of $f(x)$ is the value such that the area under $y = f_{av}$ is the same as the area under $f(x)$

this means $\underbrace{\int_a^b f_{av}\,dx}_{\text{rectangle w/ height } f_{av} \text{ and width } b-a} = \int_a^b f(x)\,dx$

$$f_{av}(b-a) = \int_a^b f(x)\,dx \rightarrow \boxed{f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

<u>example</u> $f(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$



$$f_{av} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \cos x \, dx$$

$$= \frac{2}{\pi} \left(\sin x \right) \Big|_0^{\pi/2}$$

$$= \frac{2}{\pi} \left[\sin\left(\frac{\pi}{2}\right) - \sin(0) \right] = \boxed{\frac{2}{\pi}}$$

f_{av} is not necessarily halfway between max and min

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5.5 Substitution Rule (part 1)

integration by substitution is the reverse of Chain Rule

$$f(x) = \frac{1}{4}\underbrace{(x^2+3)}_{u}{}^4 = \frac{1}{4}u^4 \qquad u = x^2+3$$

$$f' = \frac{1}{4}\cdot 4u^3 \cdot \frac{du}{dx} = u^3 \cdot 2x = (x^2+3)^3\cdot 2x$$

when we want to find the antiderivative of $(x^2+3)^3\cdot 2x$

we need to undo the chain Rule

$$\int (x^2+3)^3 \cdot (2x)\, dx$$

let $u = x^2+3$ just like in Chain Rule

then $\frac{du}{dx} = 2x$ and $\underbrace{du = 2x\, dx}_{\text{"differential of } u\text{"}}$ (multiply dx over)

$$\int \underbrace{\boxed{(x^2+3)^3}}_{u^3}\ \underbrace{\boxed{(2x)\,dx}}_{du}$$

$u = x^2+3$

$du = 2x\,dx$

$$= \int u^3\,du$$

handle this just like $\int x^3\,dx$

$$= \frac{u^4}{4} + C$$

get rid of u

$$= \boxed{\frac{1}{4}(x^2+3)^4 + C}$$

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example $\int x^2 \sqrt{x^3+4}\, dx$

$$= \int x^2 (x^3+4)^{1/2} dx$$

let $\boxed{u = x^3+4}$

why? we will find out after this example

then $\frac{du}{dx} = 3x^2$

$$\boxed{du = 3x^2 dx}$$

sub u and du into the integral, get rid of all x

$$\int \underbrace{\boxed{(x^3+4)^{1/2}}}_{u^{1/2}} \underbrace{\boxed{(x^2 dx)}}_{\frac{1}{3}du}$$

$du = 3x^2 dx$

$\underbrace{\qquad}$ in the integral

$\frac{1}{3} du = \boxed{x^2 dx}$

$$= \int u^{1/2} \cdot \frac{1}{3} du = \frac{1}{3}\int u^{1/2} du = \frac{1}{3}\left(\frac{u^{3/2}}{3/2}\right) + C$$

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$$= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C$$

Sub $u = x^3 + 4$ back

$$= \boxed{\frac{2}{9}(x^3+4)^{3/2} + C}$$

check: $\frac{d}{dx}\left[\frac{2}{9}(x^3+4)^{3/2} + C\right]$

$$= \frac{2}{9} \cdot \frac{3}{2} (x^3+4)^{1/2} \cdot (3x^2)$$

$$= \frac{1}{3} (x^3+4)^{1/2} \cdot 3x^2 = x^2\sqrt{x^3+4}$$

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How to choose u?

→ pick the part of the integrand whose derivative is a constant multiple of the other part

example 1: $\int (\boxed{x^2+3})^3 (\boxed{2x})\,dx$

deriv. of x^2+3 is $2x$, which is some multiple of the other part $(2x)$

so, we choose x^2+3 to be u

example 2: $\int \boxed{x^2} (\boxed{x^3+4})^{1/2}\,dx$

deriv. of x^3+4 is $3x^2$, which is three times of the other part (x^2)

so, we choose $u = x^3+4$

Rule of Thumb:

choose the more complicated part to be u example $\int \frac{1}{1-x}\,dx$

$$= \int (\boxed{1-x})^{-1} \cdot (\boxed{1})\,dx$$

deriv. of $1-x$ is -1, which is a multiple of the other (1)

so, we choose $\boxed{u = 1-x}$

then $\frac{du}{dx} = -1$

and $\boxed{du = -1 \cdot dx}$

$$\int \underbrace{(1-x)^{-1}}_{u^{-1}} \cdot \underbrace{(1 \cdot dx)}_{-du}$$

$-du = 1 \cdot dx$

$$= \int u^{-1} \cdot -du = -\int u^{-1}\,du = -\ln|u| + C$$

$$= \boxed{-\ln|1-x| + C}$$

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example $\int e^{2x+3}\,dx$

exponential: let the power of e be u

$\boxed{u = 2x+3}$

$\frac{du}{dx} = 2$

$\boxed{du = 2\cdot dx} \rightarrow \frac{1}{2}du = dx$

$\int e^{2x+3}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c$

$e^{2x+3} \downarrow e^u \qquad dx \downarrow \frac{1}{2}du$

$= \boxed{\frac{1}{2}e^{2x+3} + c}$

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Definite integrals involve one extra step : adjust the integration limits

<u>example</u>

$$\int_0^1 (x^2+3)^3 (2x)\, dx$$

pick u as usual : $u = x^2+3$

$$\frac{du}{dx} = 2x$$

$$du = 2x\, dx$$

now adjust the integration limits to refer to u

old upper limit : $x = 1 \rightarrow u = 1^2+3 = 4$

old lower limit : $x = 0 \rightarrow u = 0^2+3 = 3$

$$\int_3^4 u^3\, du = \underbrace{\left.\frac{u^4}{4}\right|_3^4 = \frac{4^4}{4} - \frac{3^4}{4}}_{\text{do NOT go back to } x} = \boxed{\frac{175}{4}}$$