7.2 Exponential Growth and Decay

consider a quantity that is growing or decaying at a rate that

is proportional to its size

$$\frac{dy}{dt} = k \cdot y$$

constant of proportionality ("proportional to ___")

size of the quantity

k is also called the relative growth/decay rate

for example, if something is growing at 2% of its size

then $\frac{dy}{dt} = \underbrace{0.02}_{2\%}\, y$ (for example, savings account pays 2% of your balance as interest)

ky together gives us the absolute growth/decay rate

(\$1000 in account paying 2% interest is \$20 in interest
\$10,000 " " " \$200 ")

Draw Erase Select Point Add Undo Redo Rec Stop View Close

if $\frac{dy}{dt} = ky$, what is $y(t)$?

this means the rate of change of y is some multiple of itself

only one type of function behaves this way → exponential

$$\left(\frac{d}{dt} e^{at} = e^{at} \cdot a = a e^{at} \right)$$

so, $y(t)$ must be some kind of exponential function

in form of $y = Ce^{kt}$

$$\left(\text{check: } \frac{dy}{dt} = \frac{d}{dt}\left(Ce^{kt}\right) = C \cdot e^{kt} \cdot k = \overset{\text{constant}}{k} \cdot \underbrace{Ce^{kt}}_{y} \right)$$

the constant C is the size at $t=0$

$y(0) = Ce^{k(0)} = C$ usually given the symbol y_0

so, $\boxed{\text{exponential growth/ decay: } y(t) = y_0 e^{kt} \longleftrightarrow \frac{dy}{dt} = ky}$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

if $K>0$, then relative rate is positive, so y is growing (population, interest, spread of diseases)

if $K<0$, " " negative, so y is decaying (radioactive decay, metabolism of drugs)

example The average yearly inflation rate between 2016 and 2019 in the US is 2.1%. If a loaf of bread cost \$2 in 2016, what will it cost in 2030 if the inflation rate stays the same? When will the price double?

$y(t) = y_0 e^{Kt}$

y : cost of bread (dollars)

t : time since 2016, in years

($t=0 \to 2016$, $t=1 \to 2017$, etc)

$y_0 = y(0) = 2$

2030 is 14 years after 2016

find: $y(14)$, when $y=4$, $t=?$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

$y(t) = 2e^{kt}$

now we need k

we know $y(0) = 2$, and $\underbrace{y(1) = \overbrace{1.021}^{2.1\% \text{ higher}}\, y(0) = 1.021(2) = 2.042}_{\text{plug into } y(t) = y_0 e^{kt}}$

$$\underbrace{2.042}_{y(1)} = \underbrace{2}_{y_0} e^{k \cdot 1} \leftarrow t = 1$$

$$\frac{2.042}{2} = e^{k} = 1.021$$

$$\boxed{k = \ln(1.021) = 0.0208}$$

in 2030, $t = 14$, so $y(14) = 2e^{0.0208(14)} = \boxed{2.68}$ \$2.68 in 2030

when will $y = 4$?

Draw Erase Select Point Add Undo Redo Rec Stop View Close

$$y(t) = y_0 e^{kt}$$

when $y(t) = \underbrace{2y_0}_{\text{doubling current amount}}$, $\quad 2\cancel{y_0} = \cancel{y_0} e^{kt} \rightarrow 2 = e^{kt}$

$$\ln 2 = kt$$

$$\boxed{t = \frac{\ln 2}{k}}$$ doubling time

note it does NOT depend on y_0

here, $k = 0.0208$, so bread will cost \$4 when

$$t = \frac{\ln 2}{0.0208} \approx 33 \quad (\text{years after } 2016) \rightarrow \text{in year } 2049$$

<u>example</u> The half-life of caffeine in human body is around 5 hours. If 100 mg of caffeine was ingested through a cup of coffee at 6 am, how long before 80% of the caffeine is eliminated? How much caffeine is left at 10 pm?

just like doubling time, except this is the time to <u>lose</u> half of the initial amount.

(here, 100 mg → 50 mg in 5 hours, 50 → 25 in another 5 hours and so on)

$$y(t) = y_0 e^{kt}$$

t: time since 6 am in hours

y: amount of caffeine in body, in mg

Draw Erase Select Point Add Undo Redo Rec Stop View Close

$y(t) = 100e^{kt}$

find k: half-life is 5 hours, so $\frac{1}{2}\cancel{y_0} = \cancel{y_0}e^{kt}$

$\ln\frac{1}{2} = kt$

$\boxed{t = \frac{\ln 1/2}{k}}$

half-life again, does NOT depend on y_0

we can find k from half-life:

$5 = \frac{\ln\frac{1}{2}}{k} \rightarrow k = \frac{\ln 1/2}{5} = -0.139$

relative decay rate

lose 13.9 % of the amount at any time

$y(t) = 100e^{-0.139t}$

to to lose 80% $\rightarrow$ 20% of 100 is left

$20 = 100e^{-0.139t}$

$\frac{1}{5} = e^{-0.139t}$

$$\ln \frac{1}{5} = -0.139\,t$$

$$t = \frac{\ln 1/5}{-0.139} = \boxed{11.6 \text{ hours since 6 am}}$$

at 10 pm, how much remains?

↳ $t = 16$ (hours after 6 am)

$$y(t) = 100\,e^{-0.139t}$$

$$y(16) = 100\,e^{-0.139(16)} = \boxed{10.8 \text{ mg}}$$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

<u>example</u> The E. Coli bacterium divides itself into two cells every 20 minutes. A sample initially has a population of 50 cells.

When will the population reach 1 billion?

How fast is the population growing at that time?

$y(t) = y_0 e^{kt}$

t : time in minutes

y : population of the sample

$y_0 = 50 = y(0)$

after 20 minutes, it doubles $\to$ $\underbrace{y(20) = 100}_{\text{plug into } y(t) = y_0 e^{kt}}$

$100 = 50 e^{k \cdot 20}$

$$\frac{100}{50} = e^{20k}$$

$$\ln\left(\frac{100}{50}\right) = \ln(2) = 20k \rightarrow k = \frac{\ln(2)}{20} = \underbrace{0.035}$$

relative growth rate is 3.5%

(absolute growth rate is 3.5% of the population)

time to reach pop. of 1 billion?

$$y(t) = y_0 e^{kt}$$

$$1{,}000{,}000{,}000 = 50 e^{0.035t}$$

$$20000000 = e^{0.035t}$$

$$\ln(20000000) = 0.035t$$

$$t = \frac{\ln(20000000)}{0.035} = 485 \text{ (minutes)} \approx \boxed{8 \text{ hours}}$$

absolute growth rate at that time?

$$\frac{dy}{dt} = ky = (0.035)(1{,}000{,}000{,}000) = \boxed{35{,}000{,}000 \text{ cells/min}}$$