1.3 Inverse, Exponential and Logarithmic Functions

$y = f(x)$ what is the output y when we know the input x

the <u>inverse</u> is

$x = f^{-1}(y)$ what is the input x that got me a specific output y

for example, $f(x) = x^3$

$f(2) = 2^3 = 8$ input of 2 gives output 8

$8 = x^3$

$x = 8^{1/3} = 2$ what input x gave us output 8 not all functions have inverses

a function must be <u>one-to-one</u>, at least on some portion of its domain to have an inverse.

one-to-one: each output (y) is paired with <u>at most</u> one input (x)

last example: $f(x) = x^3$

$8 = x^3 \rightarrow x = 2$ only

only possible input to get 8 is 2

so, $f(x) = x^3$ is one-to-one

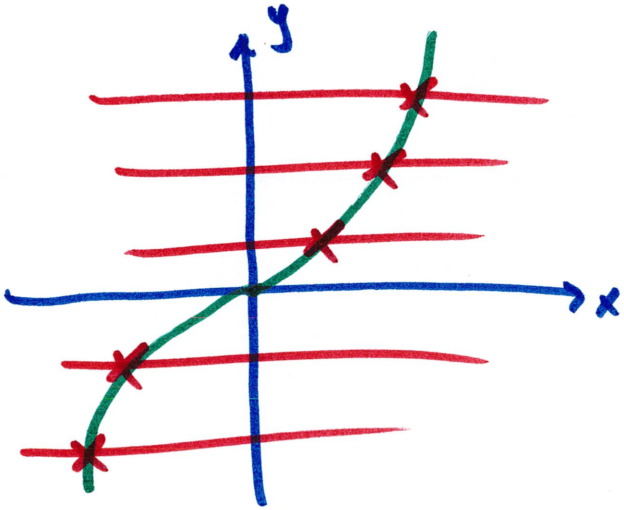
$f(x) = x^2$

is not one-to-one because output of 25

$25 = x^2 \rightarrow \underbrace{x = \pm 5}$

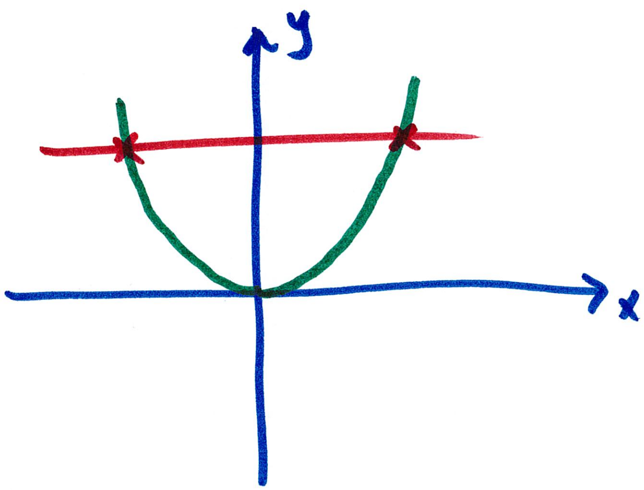
output of 25 paired with more than one input the graph of a one-to-one function must pass the

<u>horizontal line test</u> : each horizontal line drawn can only intersect the graph <u>at most once</u>



each horizontal line intersects graph at most once

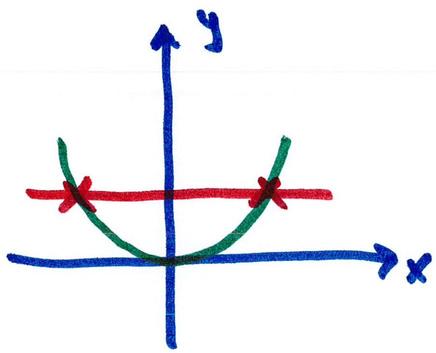
this is the graph of a one-to-one function



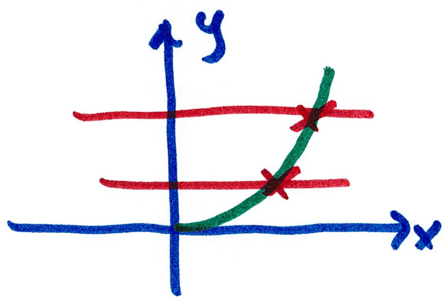
at least one horizontal line crosses the graph more than once

this is NOT the graph of a one-to-one function even for a function that is not one-to-one, we can restrict its domain to make it one-to-one

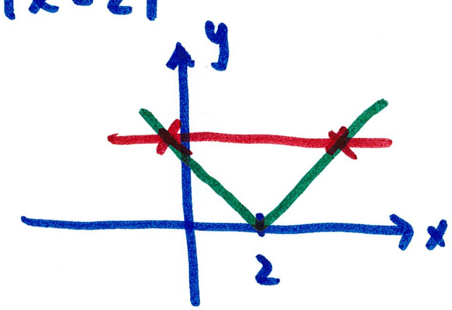
for example, $f(x) = x^2$ is not one-to-one on $(-\infty, \infty)$



but it is one-to-one if we only consider $(0, \infty)$

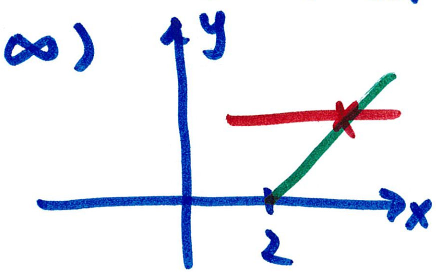


$f(x) = |x-2|$



NOT one-to-one on $(-\infty, \infty)$

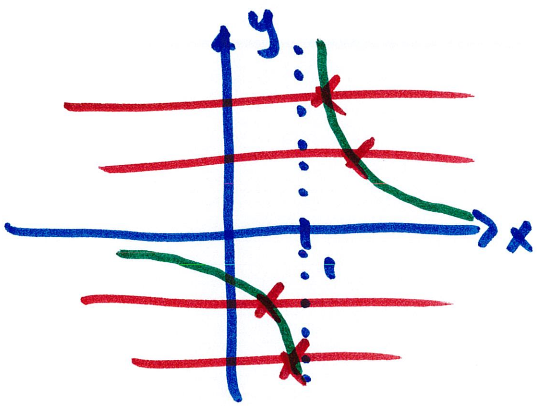
but is one-to-one on $[2, \infty)$



if $f(x)$ is one-to-one on at least some interval, then we can find its inverse on ~~the~~ some parts of it

<u>example</u> $\quad f(x) = \dfrac{1}{x-1}$

is it one-to-one?



domain: $(-\infty, 1), (1, \infty)$

passes the horizontal line test

so is one-to-one, so has an inverse

find the inverse (as an example, on $(1, \infty)$)

$$y = \frac{1}{x-1}$$

we interchange x and y (because we know output already and want the input)

$$x = \frac{1}{y-1}$$

then solve for y

multiply both sides by $y-1$ and divide by x

$$y-1=\frac{1}{x}$$

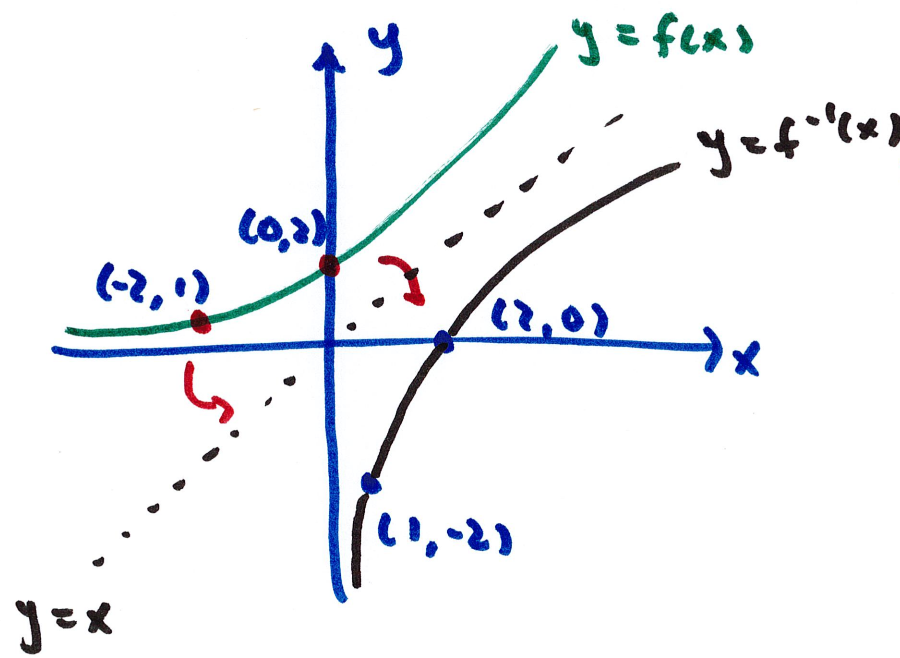
$$y=\frac{1}{x}+1$$

the rewrite y as $f^{-1}(x)$: $\boxed{f^{-1}(x)=\frac{1}{x}+1}$

$$\underbrace{0<x<\infty}$$

output is positive (right part of graph)

the relationship between graphs of $f(x)$ and $f^{-1}(x)$



x and y switch places on $f^{-1}(x)$

graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other across line $y=x$

exponential and logarithmic functions are inverses of each other

exponential		logarithmic
$y = b^x$	$\longleftrightarrow$	$x = \log_b y$
$f(x) = b^x$	$\longleftrightarrow$	$f^{-1}(x) = \log_b x$

example : $f(x) = 3^x$ $\quad f(2) = 3^2 = 9$ output is 9 when input is 2

$$f^{-1}(x) = \log_3 x$$

output

what input gave us an output of 27?

$$f^{-1}(x) = \log_3 27 = \log_3 3^3 = 3\underbrace{\log_3 3}_{1} = 3$$

useful properties of logarithm

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b x^c = c \log_b x$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$b^{\log_b x} = x \qquad \log_b b^x = x$$

for example, $\log_3 \frac{\sqrt{x}}{\sqrt[5]{y}}$

$$= \log_3 \sqrt{x} - \log_3 \sqrt[5]{y} \qquad \text{(2nd property)}$$

$$= \log_3 x^{1/2} - \log_3 y^{1/5}$$

$$= \frac{1}{2}\log_3 x - \frac{1}{5}\log_3 y \qquad \text{(3rd property)}$$

the properties can also be used to solve exponential equations

for example, $2^{5x-3} = 18 \qquad x = ?$

take log of <u>any base</u> on both sides

↳ usually base e $\quad \log_e = \ln$

or base 10 $\quad \log_{10} = \log$

$$\ln 2^{5x-3} = \ln 18$$

$$(5x-3)\cdot(\ln 2) = \ln 18$$

$$5x - 3 = \frac{\ln 18}{\ln 2} \rightarrow 5x = 3 + \frac{\ln 18}{\ln 2} \qquad \boxed{x = \frac{3}{5} + \frac{\ln 18}{5\ln 2}}$$