3.3 Rules of Differentiation

"to differentiate" – means to find the derivative of something

definition of derivative : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

geometric but not practical.

let's find pattern in derivative of x^n

$f(x) = 1 = x^0$ $\quad f'(x) = \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

$f(x) = x^1$ $\quad f'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1 = 1 \cdot x^0$

$f(x) = x^2$ $\quad f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{2x+h}{1} = 2x = 2 \cdot x^1$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$

$= \frac{1}{2} x^{-1/2}$

note the pattern: deriv. of x^n is $n x^{n-1}$

this is called the Power Rule

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

n: any real number

"derivative of ____"

(Leibniz notation)

prime (′) notation is called the Lagrange notation

build onto the power rule: what is the deriv. of $C \cdot x^n$

some constant

if $f(x) = C x^n$ since the derivative is a limit process

and we know $\lim_{x \to a} (c \cdot f(x)) = (\lim_{x \to a} c)(\lim_{x \to a} f(x))$

$= c \lim_{x \to a} f(x)$

so, $\boxed{\frac{d}{dx}(C \cdot x^n) = C \cdot \frac{d}{dx}(x^n) = C \cdot n x^{n-1}}$ C: some number in fact, $\boxed{\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x)) = c \cdot f'(x)}$

<u>Constant-Multiple Rule</u>

similarly, since $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

and since the derivative is a limit process,

$$\boxed{\begin{aligned} \frac{d}{dx}\left[f(x) + g(x)\right] &= \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \\ \frac{d}{dx}\left[f(x) - g(x)\right] &= \frac{d}{dx} f(x) - \frac{d}{dx} g(x) \end{aligned}}$$

<u>Sum/Difference Rules</u>

example $f(x) = x^{2/3} - 5\sqrt{x} + \frac{1}{2x} + 10$ $f'(x) = ?$

rewrite in the form x^n

$$f(x) = x^{2/3} - 5x^{1/2} + \frac{1}{2}x^{-1} + 10x^0$$

$$f'(x) = \frac{2}{3}x^{2/3-1} - 5\cdot\left(\frac{1}{2}x^{1/2-1}\right) + \frac{1}{2}\cdot\left(-1\cdot x^{-2}\right) + 10\cdot\left(0\cdot x^{0-1}\right)$$

$$= \frac{2}{3}x^{-1/3} - \frac{5}{2}x^{-1/2} - \frac{1}{2}x^{-2} + 0$$

$$= \boxed{\frac{2}{3x^{1/3}} - \frac{5}{2\sqrt{x}} - \frac{1}{2x^2}}$$

$f'(x)$ is itself a function, so we can evaluate it wherever it is defined

for example, $f'(1) = \frac{2}{3} - \frac{5}{2} - \frac{1}{2} = -\frac{4}{3}$

in Leibniz notation: $\left.\frac{d}{dx}\left(x^{2/3} - 5\sqrt{x} + \frac{1}{2x} + 10\right)\right|_{x=1} = -\frac{4}{3}$

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

but this is NOT true for multiplication or division

<u>example</u>

$$f(x) = (2x-7)^2$$
$$= (2x-7)(2x-7) \rightarrow$$

$$\boxed{\frac{d}{dx}\left[(2x-7)(2x-7)\right] \neq \frac{d}{dx}(2x-7) \cdot \frac{d}{dx}(2x-7)}$$

we can work around that

rewrite : $f(x) = (2x-7)(2x-7)$

$$= 4x^2 - 14x - 14x + 49$$
$$= 4x^2 - 28x + 49$$

now use sum/difference rules

$$f'(x) = 4(2x^1) - 28(1 \cdot x^0) + 49(0 \cdot x^{0-1})$$
$$= \boxed{8x - 28}$$

example $\quad f(x) = \dfrac{6x^9 - 7x^4}{5x^4}$

$$\boxed{\text{again, } f'(x) \neq \dfrac{\frac{d}{dx}(6x^9 - 7x^4)}{\frac{d}{dx}(5x^4)}}$$

workaround: rewrite as combinations of x^n

$$f(x) = \frac{6x^9}{5x^4} - \frac{7x^4}{5x^4} = \frac{6}{5}x^5 - \frac{7}{5}$$

$$f'(x) = \frac{6}{5}(5x^4) - 0 = \boxed{6x^4}$$

unfortunately, we can't work around all quotients

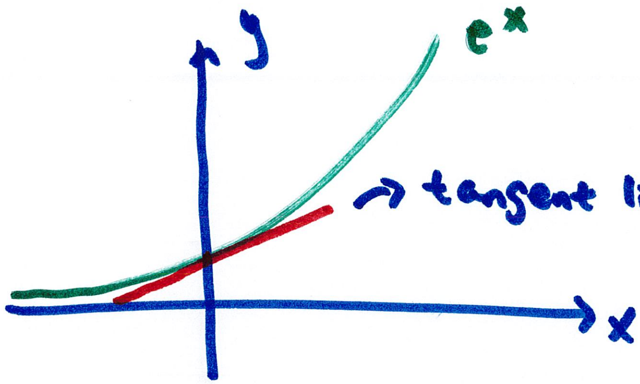
for example, if $f(x) = \dfrac{5x^4}{6x^9 - 7x^4} \neq \dfrac{5x^4}{6x^9} - \dfrac{5x^4}{7x^4}$

just like $\dfrac{5}{2+3} \neq \dfrac{5}{2} + \dfrac{5}{3}$

this is a case that requires another rule that we will see later next, let's find derivative of e^x

there are many ways to define e

one way: define e such that $f(x)=e^x$ has a tangent line with slope of 1 at $x=0$



$\rightarrow$ tangent line at $x=0$ has slope 1

this means: if $f(x)=e^x$ then $f'(0)=1$

$$f'(0)=\lim_{h\to 0}\frac{e^{0+h}-e^0}{h}=\lim_{h\to 0}\frac{e^0e^h-e^0}{h}=\boxed{\lim_{h\to 0}\frac{e^h-1}{h}=1}$$

we will come back to this

$f(x) = e^x$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right) = \left(\lim_{h \to 0} e^x \right) \underbrace{\left(\lim_{h \to 0} \frac{e^h - 1}{h} \right)}_{\text{equal to 1 from last page}}$$

$$= \lim_{h \to 0} e^x = e^x$$

this means $\boxed{\frac{d}{dx}(e^x) = e^x}$ e^x is its own derivative

example $f(x) = \dfrac{8e^{2x} + 7e^x}{e^x}$

rewrite: $f(x) = \dfrac{8e^{2x}}{e^x} + \dfrac{7e^x}{e^x} = 8e^x + 7$

$$f'(x) = 8(e^x) + 0 = \boxed{8e^x}$$

the derivative is a function, so we can take the derivative of derivative (of a derivative... of) over and over again.

$y = f(x)$

$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}$ first derivative

deriv. of y'

$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$ second deriv.

$\underbrace{y''' = f'''(x)}$ $= \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3}$ 3rd deriv.

sometimes $y^{(3)} = f^{(3)}(x)$

beyond 3, we <u>always</u> write $y^{(n)} = f^{(n)}(x)$

for example, the 15^{th} deriv. of y is $y^{(15)} = f^{(15)}(x) = \frac{d^{15}y}{dx^{15}}$