3.9 Derivatives of Logarithmic Functions

NOT on exam 2

we know $\frac{d}{dx} e^x = e^x$ and $\frac{d}{dx} e^u = e^u \frac{du}{dx}$

together with implicit differentiation, we can now find derivatives of logarithmic functions

$y = \ln x$

is equivalent to $x = e^y$

now differentiate $x = e^y$ implicitly

$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$ ← implicit function of x so chain rule applies

$1 = e^y \frac{dy}{dx}$ ← we want this

$\frac{dy}{dx} = \frac{1}{e^y}$ but $y = \ln x$

$= \frac{1}{e^{\ln x}} = \frac{1}{x}$

so, $\boxed{\frac{d}{dx} \ln x = \frac{1}{x} \quad x \neq 0,\ x > 0}$

and $\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}}$

example $y = \ln(\underbrace{\tan x}_{u})$ $\qquad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{\tan x}\frac{d}{dx}\tan x = \frac{1}{\tan x}\sec^2 x = \boxed{\frac{\sec^2 x}{\tan x}}$$

example $y = \ln(\underbrace{\ln(12x)}_{u})$

$$\frac{dy}{dx} = \frac{1}{\ln(12x)}\underbrace{\frac{d}{dx}\ln(12x)}_{\text{chain rule again}}$$

$$= \frac{1}{\ln(12x)} \cdot \frac{1}{12x} \cdot \frac{d}{dx}(12x)$$

$$= \frac{1}{\ln(12x)} \cdot \frac{1}{12x} \cdot 12 = \boxed{\frac{1}{x\ln(12x)}}$$

Example

$$y = \ln\big[\underbrace{(x^3+7)^{4\pi}}_{u}\big] \qquad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{(x^3+7)^{4\pi}} \cdot \underbrace{\frac{d}{dx}(x^3+7)^{4\pi}}_{\text{chain rule again}}$$

$$= \frac{1}{(x^3+7)^{4\pi}} \cdot 4\pi(x^3+7)^{4\pi-1} \cdot \frac{d}{dx}(x^3+7)$$

$$= \frac{1}{(x^3+7)^{4\pi}} \cdot 4\pi(x^3+7)^{4\pi-1} \cdot 3x^2$$

$$= \frac{12x^2\pi\,(x^3+7)^{4\pi-1}}{(x^3+7)^{4\pi}} = 12x^2\pi\,(x^3+7)^{-1}$$

$$= \boxed{\frac{12x^2\pi}{x^3+7}}$$

alternative way: use property of log first $\rightarrow$ $\ln a^b = b \ln a$

$y = \ln(x^3+7)^{4\pi}$ use $\ln a^b = b \ln a$

rewrite as $y = 4\pi \ln(x^3+7)$

now differentiate

$$\frac{dy}{dx} = 4\pi \cdot \frac{d}{dx} \underbrace{\ln(x^3+7)}_{\ln(u)}$$

$$= 4\pi \cdot \frac{1}{x^3+7} \cdot \frac{d}{dx}(x^3+7) = 4\pi \cdot \frac{1}{x^3+7} \cdot 3x^2 = \boxed{\frac{12x^2\pi}{x^3+7}}$$

<u>example</u>

$$y = \ln\left(\frac{x+1}{x-1}\right)$$

option 1: $y = \ln\underbrace{\left(\frac{x+1}{x-1}\right)}_{u}$ $\qquad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{\frac{x+1}{x-1}} \cdot \underbrace{\frac{d}{dx}\left(\frac{x+1}{x-1}\right)}_{\text{quotient rule}} = \frac{x-1}{x+1} \cdot \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$= \frac{1}{x+1} \cdot \frac{-2}{x-1} = \boxed{\frac{-2}{(x+1)(x-1)}}$$

option 2: log property $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$$y = \ln\left(\frac{x+1}{x-1}\right)$$

rewrite

$$y = \ln(x+1) - \ln(x-1)$$

$$\frac{dy}{dx} = \frac{1}{x+1} \cdot \frac{d}{dx}(x+1) - \frac{1}{x-1} \cdot \frac{d}{dx}(x-1)$$

$$= \boxed{\frac{1}{x+1} - \frac{1}{x-1}}$$

is it equal to $\frac{-2}{(x+1)(x-1)}$?

$$\frac{1}{x+1} \cdot \frac{x-1}{x-1} - \frac{1}{x-1} \cdot \frac{x+1}{x+1} = \frac{(x-1)-(x+1)}{(x+1)(x-1)} = \frac{-2}{(x+1)(x-1)}$$

In general, it is always easier to handle more simpler log functions than to deal with fewer but more complicated log functions we can take advantage of log properties even if ~~the~~ logs don't appear initially

→ <u>Logarithmic Differentiation</u>

<u>example</u> $y = (x^4+2)^3 (x^5+1)^6$

option 1: leave as is, use product + chain rules

$$\frac{dy}{dx} = (x^4+2)^3 \cdot \frac{d}{dx}\underbrace{(x^5+1)^6}_{u^n} + (x^5+1)^6 \cdot \frac{d}{dx}\underbrace{(x^4+2)^3}_{u^n}$$

← chain rule →

$$= (x^4+2)^3 \cdot 6(x^5+1)^5 \cdot 5x^4 + (x^5+1)^6 \cdot 3(x^4+2)^2 \cdot 4x^3$$

$= \ldots$ (simplify)

option 2: logarithmic differentiation

$$\boxed{y = (x^4+2)^3 (x^5+1)^6}$$

take ln on both sides

$$\ln y = \ln\left[(x^4+2)^3 (x^5+1)^6\right]$$

$\ln(ab) = \ln(a) + \ln(b)$

$$= \ln(x^4+2)^3 + \ln(x^5+1)^6$$

$\ln(a^b) = b \ln a$

$$\ln y = 3\ln(x^4+2) + 6\ln(x^5+1)$$

make whatever following ln as simple as you can

differentiate implicitly, solve for $\frac{dy}{dx}$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[3\ln(x^4+2)\right] + \frac{d}{dx}\left[6\ln(x^5+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3\cdot\frac{1}{x^4+2}\cdot 4x^3 + 6\cdot\frac{1}{x^5+1}\cdot 5x^4$$

want this! →

$$\frac{1}{y}\frac{dy}{dx} = \underbrace{\frac{12x^3}{x^4+2} + \frac{30x^4}{x^5+1}}$$

$$\frac{dy}{dx} = y\left(\quad\downarrow\quad\right) = \boxed{(x^4+2)^3(x^5+1)^6\left(\frac{12x^3}{x^4+2} + \frac{30x^4}{x^5+1}\right)}$$

example $y = x^x$ $\qquad \frac{dy}{dx} = ?$ $\qquad \frac{d}{dx} x^n = n x^{n-1}$ only if exponent is a constant

take ln on both sides

$\ln y = \ln x^x$ $\qquad \ln(a^b) = b \ln a$

$\ln y = (x)(\ln x)$

differentiate implicitly

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot (1)$ product rule

$= 1 + \ln x$

$\frac{dy}{dx} = y(1+\ln x) = \boxed{x^x(1+\ln x)}$

we know $\frac{d}{dx} e^x = e^x$ but $\frac{d}{dx} a^x \neq a^x$ if $a \neq e$

e.g $\frac{d}{dx} 2^x \neq 2^x$

but what is it?

$y = a^x$ $\frac{dy}{dx} = ?$

use logarithmic differentiation

$\ln y = \ln a^x$

$\ln y = x \cdot \ln a = \underbrace{(\ln a)}_{\text{constant}} x$

differentiate implicitly

$\frac{1}{y} \frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a = a^x \ln a$

check: if $a = e$ does it match what we know?

$\frac{d}{dx} a^x = a^x \ln a$ if $a = e$

then $\frac{d}{dx} e^x = e^x \underbrace{\ln e}_{1}$

$\frac{d}{dx} e^x = e^x$

$$\boxed{\frac{d}{dx} a^x = a^x \ln a}$$

similarly, $\boxed{\frac{d}{dx} \log_a x = \frac{1}{x \ln a}}$