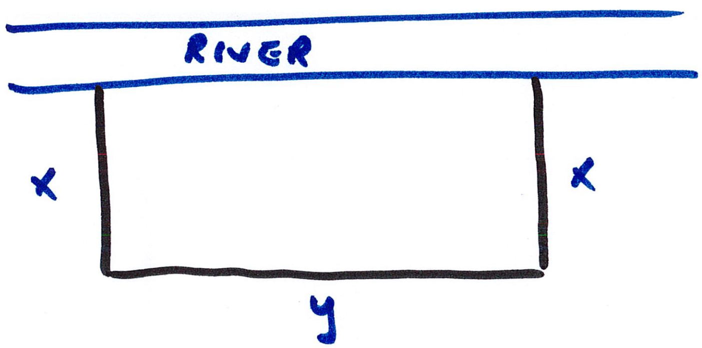
4.5 Optimization Problems (part 1)

example A farm is set up next to a river. Fencing is not required along the river. If 400 m of fencing is available to fence the other 3 sides. Find the dimensions of the farm w/ max area.



how long should the width and length be so area is max while fencing 400 m?

x: width

y: length

goal: maximize area $A = xy$ objective function (the quantity to be max/min)

condition: $2x + y = 400$ (fencing available) constraint function (condition the variables must meet)

$A = xy$ normally, we find A' and critical numbers, etc.

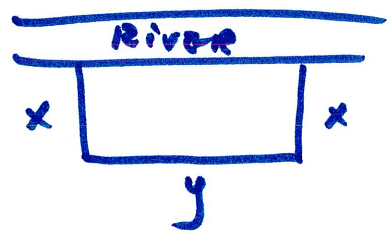
↳ has two variables

must eliminate one of them by using the constraint function

$2x + y = 400 \longrightarrow y = 400 - 2x$ sub into objective

$A = x(400 - 2x)$

$\boxed{A = 400x - 2x^2}$ objective w/ one variable



domain: $0 \leq x \leq 200$

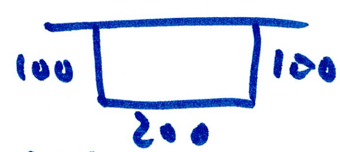
↳ if all 400 m used on width

critical numbers: $A' = 400 - 4x = 0$ $\boxed{x = 100}$

compare $A(0) = 0$

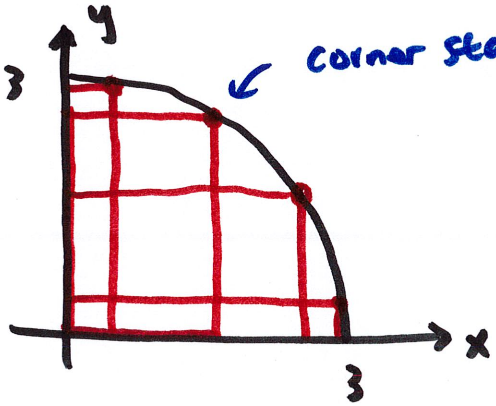
$A(100) = 400(100) - 2(100)^2 = 20{,}000 \rightarrow$ max

$A(200) = 400(200) - 2(200)^2 = 0$



to maximize A, we want $x = 100$, $y = 400 - 2x = 400 - 2(100) = 200$

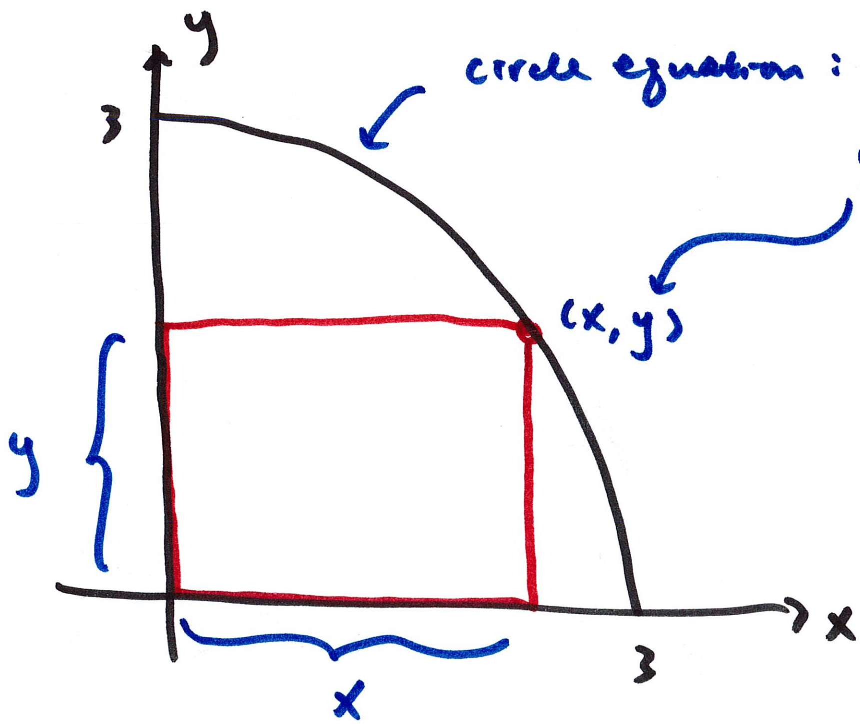
example what is the area of the largest rectangle that can be inscribed inside the first quadrant ~~portio~~ portion of a circle radius 3 ?



corner stays on circle

where should the corner be to max area of rectangle?

x : length y : height



circle equation : $x^2+y^2=9$

$y=\sqrt{9-x^2}$

area of rectangle

$$A = xy = x\sqrt{9-x^2}$$

we want to maximize $A = x\sqrt{9-x^2}$, $0 \le x \le 3$

find critical numbers: $A' = \cdots = \dfrac{9x-2x^3}{(9x^2-x^4)^{1/2}}$

$A' = 0 \to 9x - 2x^3 = 0$

$x(9-2x^2) = 0$

$\boxed{x=0}$, $x^2 = 9/2$

$\boxed{x = \cancel{\frac{\sqrt{3}}{2}}\ \frac{3}{\sqrt{2}}}$, $x = \cancel{-}\ \cancel{\frac{\sqrt{3}}{2}}\ -\frac{3}{\sqrt{2}}$

outside the domain

A' DNE $\to 9x^2 - x^4 = 0$

$x^2(9-x^2) = 0$

$\boxed{x=0}$ $x^2 = 9$

$\boxed{x=3}$, $x \cancel{=-}3$

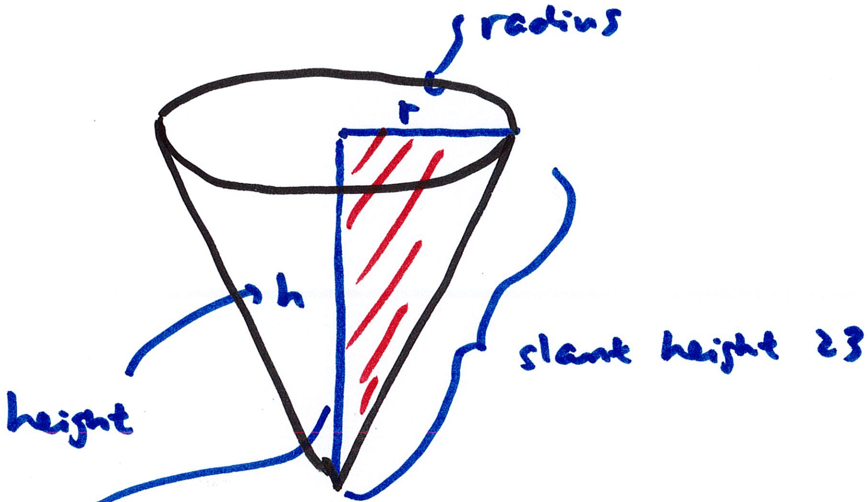
compare: $A(0) = 0$

$\cancel{A\left(\frac{\sqrt{3}}{2}\right)} = A\left(\frac{3}{\sqrt{2}}\right) = \frac{9}{2}$

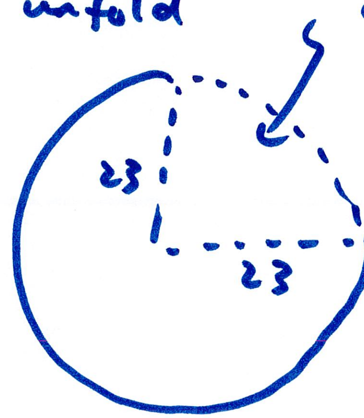
$A(3) = 0$

area of largest possible rectangle inscribed is 9/2

<u>example</u> Find the radius and the height of the largest possible cone (max volume) with a slant height of 23.



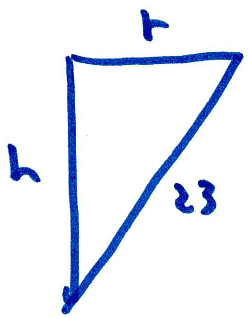
cut and unfold



amount cut out changes radius and height of resulting cone

volume of cone: $V = \frac{1}{3}\pi r^2 h$ → like in first example, this object function has two variables r and h

to eliminate r or h, we need to relate them



Pythagorean Theorem: $(23)^2 = r^2 + h^2 = 529$

$r^2 = 529 - h^2$

sub r^2 out in objective

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(529 - h^2)(h) = \frac{529}{3}\pi h - \frac{1}{3}\pi h^3$$

find critical #s : $V' = \frac{529}{3}\pi - \pi h^2 = 0$

$$h^2 = \frac{529}{3} \qquad h = \sqrt{\frac{529}{3}}$$

this is a critical number which is possibly the location of a max or min

verify w/ either First Derivative Test or Second Derivative Test

here, let's use Second Deriv. Test

$$V'' = -2\pi h$$

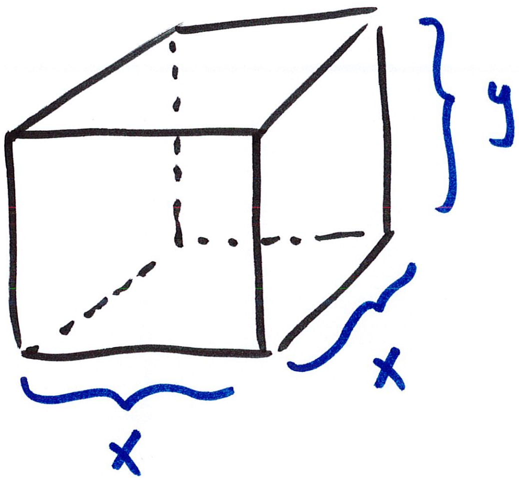
$V''\left(\sqrt{\frac{529}{3}}\right) < 0 \rightarrow$ so $\boxed{h = \sqrt{\frac{529}{3}}}$ maximizes the cone volume

radius: $r^2 = 529 - h^2 = 529 - \frac{529}{3} = \frac{2(529)}{3}$

$$\boxed{r = \sqrt{\frac{2(529)}{3}}}$$

__example__ we want to build a storage box with a square base and a volume of ~~10~~ ~~4~~ 2 ft^3. we don't want a lid.

If the material for the base costs \$3 / ft^2 and for the sides costs \$2 / ft^2. what dimensions minimizes the cost?



constraints: $x^2y = 2$ (volume)

we want to minimize cost

$$C = \underbrace{(3)}_{\text{cost per ft}^2\text{ for base}}\underbrace{(x^2)}_{\text{area of base}} + \overset{\text{four sides}}{(4)}\underbrace{(2)}_{\text{per ft}^2\text{ cost for side}}\underbrace{(xy)}_{\text{area of side}} = 3x^2 + 8xy \quad \text{objective}$$

from $x^2y=2 \rightarrow y=\frac{2}{x^2}$

Sub into C

$$C = 3x^2+8x\left(\frac{2}{x^2}\right) = 3x^2+\frac{16}{x}$$

$$C' = 6x - \frac{16}{x^2} = 0 \rightarrow x = \sqrt[3]{\frac{8}{3}}$$ critical pt

use First / Second Deriv. Test to verify that C is a minimum

$$C'' = 6 + \frac{32}{x^3}$$

$C''\left(\sqrt[3]{\frac{8}{3}}\right) > 0 \rightarrow$ C is minimized when $x = \sqrt[3]{\frac{8}{3}}$

$$y = \frac{2}{x^2} = \frac{2}{\left(\sqrt[3]{\frac{8}{3}}\right)^2}$$