2.5 Limits at Infinity

(NOT on exam 1)

$f(x) = \frac{1}{x}$ $\quad \lim_{x \to a} \frac{1}{x}$ is easy if a is a number

if a goes to a very large positive number $\rightarrow \lim_{x \to \infty} f(x)$

if a goes to a very large negative number $\rightarrow \lim_{x \to -\infty} f(x)$

$\lim_{x \to \infty} \frac{1}{x} \rightarrow \frac{1}{\text{very large number}} \rightarrow$ very small number (practically but not exactly zero)

so, we see $\boxed{\lim_{x \to \infty} \frac{1}{x} = 0}$ we can make $\frac{1}{x}$ as close to 0 as we want by making x sufficiently large

likewise, we see $\boxed{\lim_{x \to -\infty} \frac{1}{x} = 0}$

some constant (number)

$$\lim_{x\to\pm\infty} \frac{c}{x} = \lim_{x\to\pm\infty} c\cdot\frac{1}{x} = \underbrace{\left(\lim_{x\to\pm\infty} c\right)}_{c}\underbrace{\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)}_{0} = 0$$

Similarly, $$\lim_{x\to\pm\infty}\frac{c}{x^n} = \left(\lim_{x\to\pm\infty} c\right)\left(\lim_{x\to\pm\infty}\frac{1}{x^n}\right)$$

$$= \underbrace{\left(\lim_{x\to\pm\infty} c\right)}_{c}\underbrace{\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^n}_{0} = 0$$

the higher the n, the faster $\frac{c}{x^n}\to 0$

important result: $$\boxed{\lim_{x\to\pm\infty}\frac{c}{x^n} = 0} \quad (n>0)$$

↳ using this we can handle just about rational function

example

$$\lim_{x\to\infty} \frac{4x^2-x+1}{\boxed{3x^2}+5x-3}$$

first, identify the <u>dominant term</u> in <u>denominator</u>

(which is usually the term w/ highest power)

note its power

here, it's $3x^2$, its power is 2

next, divide top and bottom by x to that power

in this case, we divide top and bottom by x^2

$$\lim_{x\to\infty} \frac{\frac{4x^2-x+1}{x^2}}{\frac{3x^2+5x-3}{x^2}} = \lim_{x\to\infty} \frac{\frac{4x^2}{x^2}-\frac{x}{x^2}+\frac{1}{x^2}}{\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{3}{x^2}}$$

$$= \lim_{x\to\infty} \frac{4-\overset{\to 0}{\left(\frac{1}{x}\right)}+\overset{\to 0}{\left(\frac{1}{x^2}\right)}}{3+\underset{\to 0}{\left(\frac{5}{x}\right)}-\underset{\to 0}{\left(\frac{3}{x^2}\right)}} = \boxed{\frac{4}{3}}$$

as x becomes larger and larger $\frac{4x^2-x+1}{3x^2+5x-3}$ becomes closer and closer to $4/3$

we handle $x \to -\infty$ exactly the same way

$$\lim_{x\to-\infty} \frac{4x^2-x+1}{3x^2+5x-3} = \lim_{x\to-\infty} \frac{\frac{4x^2-x+1}{x^2}}{\frac{3x^2+5x-3}{x^2}} = \lim_{x\to-\infty} \frac{4-\overset{\to 0}{\frac{1}{x}}+\overset{\to 0}{\frac{1}{x^2}}}{3+\underset{\to 0}{\frac{5}{x}}-\underset{\to 0}{\frac{3}{x^2}}} = \frac{4}{3}$$

for rational functions (polynomial over polynomial) the limits at ∞ and $-\infty$ are the same.

<u>example</u> $\lim_{x\to\infty} \frac{x-7}{x^2+9}$

dominant term in denominator: x^2 power is 2
so we divide top and bottom by x^2

$$\lim_{x\to\infty} \frac{\frac{x-7}{x^2}}{\frac{x^2+9}{x^2}} = \lim_{x\to\infty} \frac{\frac{x}{x^2}-\frac{7}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}} = \lim_{x\to\infty} \frac{\overset{\to 0}{\frac{1}{x}}-\overset{\to 0}{\frac{7}{x^2}}}{1+\underset{\to 0}{\frac{9}{x^2}}} = \frac{0}{1} = 0$$

example

$$\lim_{x\to\infty} \frac{x^3+7x^2}{x+9}$$

dominant term in denominator : x (power is 1)

so divide top and bottom by x^1

$$\lim_{x\to\infty} \frac{\frac{x^3+7x^2}{x}}{\frac{x+9}{x}} = \lim_{x\to\infty} \frac{\overset{\to\infty}{x^2} + \overset{\to\infty}{7x}}{1 + \underset{\to 0}{\frac{9}{x}}} = \frac{\infty}{1} = \infty$$

Pattern

If **numerator** has **greater** ~~power~~ degree, limit is ∞ or $-\infty$ as $x \to \pm\infty$

If **denominator** has **greater** degree, limit is 0 as $x \to \pm\infty$

If degrees are **the same** limit as $x \to \pm\infty$ is the ratio of the coefficients of terms w/ highest power

$$\lim_{x\to\infty} \frac{\boxed{4}x^2 - x + 1}{\boxed{3}x^2 + 5x - 3} = \frac{4}{3}$$

graphically, $\lim_{x \to \infty} f(x)$ is the <u>horizontal asymptote</u> to the <u>RIGHT</u>

$\lim_{x \to -\infty} f(x)$ is the <u>horizontal asymptote</u> to the <u>LEFT</u>

for rational functions, these will always be the same

but if root functions are involved, sometimes $\lim_{x \to \infty} f(x) \neq \lim_{x \to -\infty} f(x)$

for example, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\sqrt{x^2} = |x|$

so, $\lim_{x \to \infty} \sqrt{x^2} = \lim_{x \to \infty} |x| = \infty$ because $x \to \infty$ means $x \geq 0$

$\lim_{x \to -\infty} \sqrt{x^2} = \lim_{x \to -\infty} |x| = -\infty$

example

$$f(x) = \frac{3x+2}{x+\sqrt{4x^2+1}}$$

$$\lim_{x\to-\infty} f(x) = ?$$

dominant term in denominator : x or $\underbrace{\sqrt{4x^2+1}}$

$\approx \sqrt{4x^2}$ as $x \to -\infty$

$\approx 2x$

both have power of 1

so divide top and bottom by x^1

$$\lim_{x\to-\infty} \frac{3x+2}{x+\sqrt{4x^2+1}} = \lim_{x\to-\infty} \frac{\frac{3x+2}{x}}{\frac{x+\sqrt{4x^2+1}}{x}} = \cancel{\lim_{x\to-\infty} \frac{\frac{3x+2}{x}}{}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{2}{x}}{\frac{x}{x}+\frac{\sqrt{4x^2+1}}{-\sqrt{x^2}}}$$

bottom of last page:

$$\lim_{x\to-\infty} \frac{\frac{3x}{x}+\frac{2}{x}}{\frac{x}{x}+\underbrace{\frac{\sqrt{4x^2+1}}{-\sqrt{x^2}}}} = \lim_{x\to-\infty} \frac{3+\frac{2}{x}}{1-\sqrt{\frac{4x^2+1}{x^2}}} = \lim_{x\to-\infty} \frac{3+\frac{2}{x}}{1-\sqrt{4+\frac{1}{x^2}}}$$

$\frac{2}{x} \to 0$, $\frac{1}{x^2} \to 0$

needs negative because $x \to -\infty$ and $\sqrt{x^2} = |x|$

$$= \frac{3}{1-\sqrt{4}} = \frac{3}{1-2}$$

$$= -3$$