8.3 Trig Integrals (part 1)

NOT on exam 1

integrals involving cosine, sine, tangent, secant

$$\int \frac{\sin x}{\cos x}\,dx$$

$u = \cos x \qquad du = -\sin x\,dx$

$$= \int \frac{-1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C$$

basic idea: remember $\cos x$ and $\sin x$ are related by derivative

→ substitution possible if they both show up.

<u>example</u>

$$\int \frac{1}{1-\sin x}\,dx$$

can't integrate directly. bring $\cos x$ in somehow?

$$= \int \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx$$

$$= \int \frac{1+\sin x}{1-\sin^2 x}\,dx$$

$\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1-\sin^2 x$

$$= \int \frac{1+\sin x}{\cos^2 x}\,dx = \underbrace{\int \frac{1}{\cos^2 x}\,dx}_{\sec^2 x} + \underbrace{\int \frac{\sin x}{\cos^2 x}\,dx}_{u=\cos x \quad du=-\sin x\,dx}$$

$$= \int \sec^2 x\,dx - \int \frac{1}{u^2}\,du$$

$$= \tan x + \frac{1}{u} + C = \boxed{\tan x + \frac{1}{\cos x} + C}$$

<u>example</u> $\displaystyle\int_{-\pi/2}^{0} \sqrt{1+\cos(2x)}\,dx$

somehow bring sine into this

$$= \int_{-\pi/2}^{0} \sqrt{\frac{1+\cos(2x)}{1} \cdot \frac{1-\cos(2x)}{1-\cos(2x)}}\,dx$$

$$= \int_{-\pi/2}^{0} \sqrt{\frac{1-\cos^2(2x)}{1-\cos(2x)}}\,dx = \int_{-\pi/2}^{0} \sqrt{\frac{\sin^2(2x)}{1-\cos(2x)}}\,dx$$

$$= \int_{-\pi/2}^{0} \frac{|\sin(2x)|}{\sqrt{1-\cos(2x)}}\,dx \qquad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

interval: $-\pi/2 \leq x \leq 0$

$\sin(2x) < 0$ on this interval

so, $|\sin(2x)| = -\sin 2x$

$$\int_{-\pi/2}^{0} \frac{-\sin(2x)}{\sqrt{1-\cos(2x)}}\,dx$$

$$u = 1-\cos(2x)$$

$$du = 2\sin(2x)\,dx$$

$$\vdots$$

$$\dots = \boxed{\sqrt{2}}$$

example

$$\int \sin^2 x \cos^5 x \, dx$$

basic idea: let $u = \cos x$ or $u = \sin x$ and see which works

strategy for $\int \sin^m x \cos^n x \, dx$

case 1: if m or n is positive and odd

then split one power of the part w/ odd power and save it, then use $\sin^2 x + \cos^2 x = 1$ to turn everything into the other one

case 2: if m and n are both positive and even

then use $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$

back to $\int \sin^2 x \cos^5 x \, dx$

here $\cos x$ has positive odd power

Save a factor of $\cos x$

$$\int \sin^2 x \cos^4 x \cos x \, dx$$

now turn everything into $\sin x$

$$\downarrow$$

$$(\cos^2 x)^2 = (1-\sin^2 x)^2$$

$$= \int \sin^2 x (1-\sin^2 x)^2 \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

the factor of $\cos x$ saved

$$= \int u^2 (1-u^2)^2 \, du$$

$$= \dots = \boxed{\frac{1}{3}\sin^3 x - \frac{3}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$$

example

$$\int \sin^2 x \cos^2 x \, dx$$

both even power : use $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$$= \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx$$

$$= \int \frac{1-\cos^2(2x)}{4} \, dx = \frac{1}{4} \int 1-\cos^2(2x) \, dx$$

$$= \frac{1}{4} \int \sin^2(2x) \, dx$$

use $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$\sin^2(2x) = \dfrac{1-\cos(2(2x))}{2}$

$= \dfrac{1-\cos(4x)}{2}$

$$= \frac{1}{4} \int \frac{1-\cos(4x)}{2} \, dx$$

$$= \frac{1}{8} \int 1-\cos(4x) \, dx$$

$$= \boxed{\frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right) + C}$$

now brief look at $\tan x$ and $\sec x$

basic idea: substitution knowing $\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

and $\tan^2 x + 1 = \sec^2 x$

<u>example</u> $\int \tan^3 x \, dx$

we see $\tan x$, we would like to have $\sec^2 x$ somewhere

$$\int \tan x \cdot \underbrace{\tan^2 x}_{\sec^2 x - 1} dx = \int \tan x (\sec^2 x - 1) dx$$

$$= \underbrace{\int \tan x \cdot \sec^2 x \, dx}_{\substack{u = \tan x \\ du = \sec^2 x \, dx}} - \underbrace{\int \tan x \, dx}_{\tan x = \frac{\sin x}{\cos x}} = \dots = \boxed{\frac{1}{2}\tan^2 x + \ln|\cos x| + C}$$

$u = \cos x$ $du = -\sin x \, dx$
(see first example)

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\csc x = \cancel{-\csc x \cot}$$

$$= -\csc x \cot x$$

$$1 + \cot^2 x = \csc^2 x$$

<u>example</u>

$$\int \cot^2 x \, dx$$

$$= \int (\csc^2 x - 1)\, dx$$

$$= \int \csc^2 x \, dx - \int dx$$

$$= \boxed{-\cot x - x + C}$$

SI EXAM REVIEW

MON. 9/18 5:30 - 7:30 pm

MATH 175