8.4 Trigonometric Substitution (part 1)

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

trig substitution: x = some trig function

let $x = \sin\theta$ → why? we'll see in the following example

$dx = \cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta = \int \frac{1}{\sqrt{\cos^2\theta}}\cos\theta\, d\theta$$

$$= \int \frac{1}{\cos\theta}\cos\theta\, d\theta = \int d\theta = \theta + C$$

now back to x

$x = \sin\theta \rightarrow \theta = \sin^{-1}(x)$

$$= \boxed{\sin^{-1}(x) + C}$$

example

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

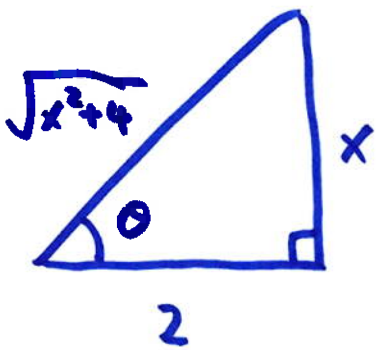
to determine the right trig subs, look at the radical part

$$\sqrt{x^2+4}$$

set up a triangle with sides $\sqrt{x^2+4}$, x, 2

square root

square root



hypotenuse : $\sqrt{x^2+4}$ because

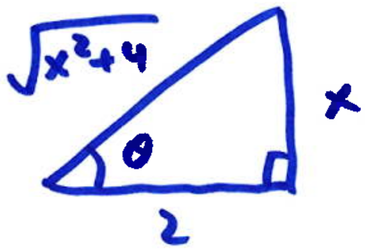
$$\left(\sqrt{x^2+4}\right)^2 = x^2 + 2^2$$

other two sides

two remaining sides : x, 2

rule of thumb : place constant as adjacent relate x and θ in the simplest way using the triangle

↳ use the simpler sides



simplest way : $\tan = \frac{x}{2}$

$$\boxed{x = 2\tan\theta}$$

$$dx = 2\sec^2\theta \, d\theta$$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{(2\tan\theta)^3}{\sqrt{(2\tan\theta)^2+4}} \cdot 2\sec^2\theta \, d\theta$$

$$= \int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta \, d\theta$$

$\tan^2\theta + 1 = \sec^2\theta$

↳ $\sqrt{4(\tan^2\theta+1)} = 2\sqrt{\sec^2\theta} = 2\sec\theta$

$$= \int \frac{8\tan^3\theta}{\cancel{2}\,\cancel{\sec}\theta} \cdot \cancel{2}\sec^{\cancel{2}}\theta \, d\theta = 8\int \tan^3\theta \sec\theta \, d\theta$$

$$= 8 \int \tan^2\theta \, \sec\theta \tan\theta \, d\theta$$

$u = \sec\theta$

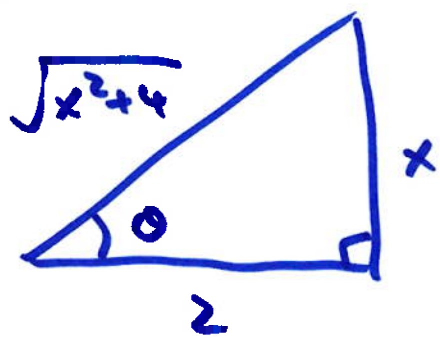
$du = \sec\theta \tan\theta \, d\theta$

$$= 8 \int (\sec^2\theta - 1) \sec\theta \tan\theta \, d\theta$$

$$= 8 \int (u^2 - 1) \, du = 8\left(\frac{u^3}{3} - u\right) + C = \frac{8}{3}\sec^3\theta - 8\sec\theta + C$$

not done yet, go back to x

bring triangle back:



$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2+4}}{2}$$

$$= \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\left(\frac{\sqrt{x^2+4}}{2}\right) + C = \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4(x^2+4)^{1/2} + C}$$

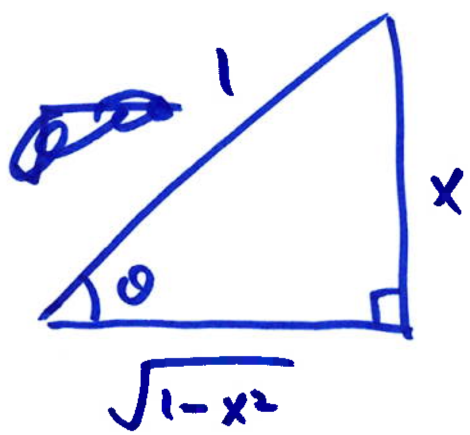
example

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{(\sqrt{1-x^2})^3}\,dx$$

construct triangle w/ sides : $\sqrt{1-x^2}$, 1 , x

root root

determine hypotenuse : 1 , because $1^2 = (\sqrt{1-x^2})^2 + x^2$



remaining sides : $\sqrt{1-x^2}$, x

neither is constant, so we take the one containing a constant as adjacent relate x and θ using simplest sides: $\sin\theta = \frac{x}{1}$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{(1-\sin^2\theta)^{3/2}} \cos\theta \, d\theta$$

$$= \int \frac{1}{(\cos^2\theta)^{3/2}} \cos\theta \, d\theta = \int \frac{1}{\cos^3\theta} \cos\theta \, d\theta = \int \frac{1}{\cos^2\theta} \, d\theta$$

$$= \int \sec^2\theta \, d\theta = \tan\theta + C$$

go back to x:

$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$

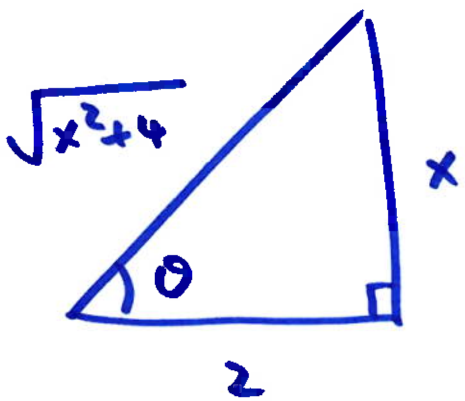
$\tan\theta = \frac{opp}{adj} = \frac{x}{\sqrt{1-x^2}}$

__example__

$$\int_0^2 \frac{x^2}{x^2+4}\,dx = \int_0^2 \frac{x^2}{\left(\sqrt{x^2+4}\right)^2}\,dx$$

triangle w/ sides: $\sqrt{x^2+4}$, x, 2

hypotenuse : $\sqrt{x^2+4}$ adjacent : 2



$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

now change integration limits to refer to θ

upper limit : $x=2 \longrightarrow \theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

lower limit : $x=0 \rightarrow \theta = \tan^{-1}\left(\frac{0}{2}\right) = \tan^{-1}(0) = 0$

$$\int_0^2 \frac{x^2}{x^2+4}\,dx = \int_0^{\pi/4} \frac{(2\tan\theta)^2}{(2\tan\theta)^2+4}\cdot 2\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{4\tan^2\theta}{4(\tan^2\theta+1)}\cdot 2\sec^2\theta\,d\theta = \int_0^{\pi/4} \frac{\cancel{4}\tan^2\theta}{\cancel{4}\,\cancel{\sec^2\theta}}\cdot 2\,\cancel{\sec^2\theta}\,d\theta$$

$$= 2\int_0^{\pi/4} \tan^2\theta\,d\theta = 2\int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta$$

$$= 2\left(\int_0^{\pi/4} \sec^2\theta\,d\theta - \int_0^{\pi/4} d\theta\right)$$

$$= 2\left(\tan\theta\Big|_0^{\pi/4} - \theta\Big|_0^{\pi/4}\right) = 2\left(1-0-\frac{\pi}{4}-0\right) = \boxed{2\left(1-\frac{\pi}{4}\right)}$$