

8.4 Trig Subs (part 2)

Example Find the length of $y = \sqrt{1-x^2}$ from $x=0$ to $x = \frac{1}{\sqrt{2}}$

$$L = \int_a^b \sqrt{1+(y')^2} dx$$

$$y = (1-x^2)^{1/2}$$

$$y' = \frac{1}{2} (1-x^2)^{-1/2} (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$1+(y')^2 = 1 + \left(\frac{-x}{\sqrt{1-x^2}} \right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

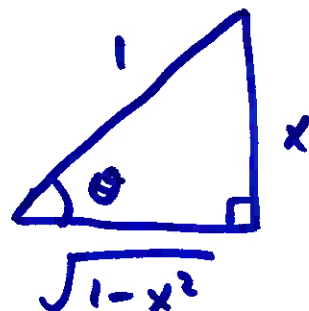
$$L = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{\frac{1}{1-x^2}} dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}} dx$$

now knowing trig subs
we can evaluate this

triangle with sides $\sqrt{1-x^2}$, 1, x

hypotenuse: 1

adjacent: $\sqrt{1-x^2}$ (part containing a constant)



relate x and θ using simplest sides:

$$\sin \theta = \frac{x}{1}$$

$$x = \sin \theta \rightarrow \theta = \sin^{-1}(x)$$

$$dx = \cos \theta d\theta$$

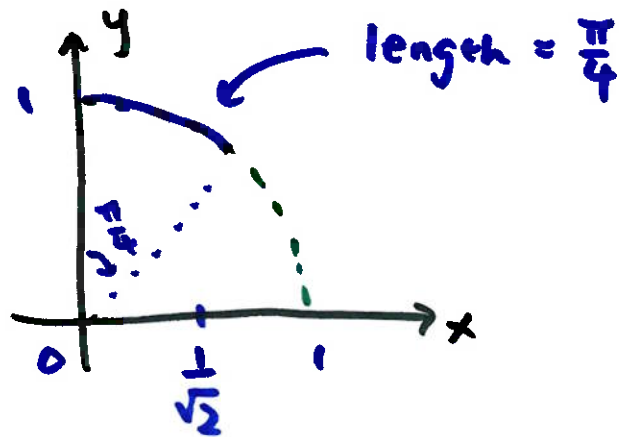
upper limit: $x = \frac{1}{\sqrt{2}} \rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

lower limit: $x = 0 \rightarrow \theta = \sin^{-1}(0) = 0$

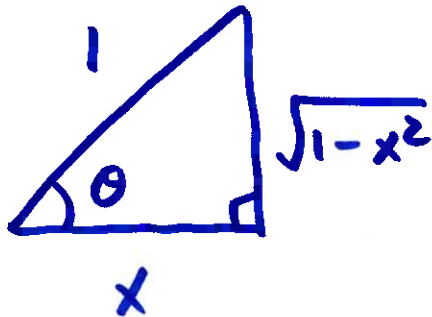
$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\underbrace{\sqrt{1-\sin^2 \theta}}_{\cos \theta}} \cos \theta d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\cos \theta} \cos \theta d\theta = \int_0^{\frac{\pi}{4}} d\theta = \theta \Big|_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{4}}$$

geometrically, this was part of the circumference of a circle radius 1



we cannot have other hypotenuse, both adj and opp can be swapped



relate x and θ : $\cos \theta = \frac{x}{1}$

$$x = \cos \theta \rightarrow \theta = \cos^{-1}(x)$$

$$dx = -\sin \theta d\theta$$

$$x = \frac{1}{\sqrt{2}} \rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$x = 0 \rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$$

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}} dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{\sqrt{1-\cos^2 \theta}} \cdot -\sin \theta d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{\sin \theta} \cdot -\sin \theta d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -d\theta = -\theta \bigg|_{\frac{\pi}{2}}^{\frac{\pi}{4}} = -\frac{\pi}{4} + \frac{\pi}{2} = \boxed{\frac{\pi}{4}}$$

the rule of thumb of putting constant or constant-containing part as adjacent limits the subs to 3: $\sin \theta$, $\sec \theta$, $\tan \theta$
 swapping adj and opp brings in additional 3: $\cos \theta$, $\csc \theta$, $\cot \theta$

example

$$\int \frac{1}{x^2 - 6x + 45} dx = \int \frac{1}{(\sqrt{x^2 - 6x + 9 + 5})^2} dx$$

notice the part under radical doesn't look like a sum or difference of squares $\rightarrow$ useful for building triangles

complete the square : $x^2 - 6x + 45$

$$= x^2 - 6x + \left(\frac{-6}{2}\right)^2 + 45 - \left(\frac{-6}{2}\right)^2$$

$$= x^2 - 6x + 9 + 45 - 9$$

$$= (x - 3)^2 + 6^2$$

$$\int \frac{1}{(\sqrt{(x-3)^2 + 6^2})^2} dx$$

$$\int \frac{1}{(\sqrt{(x-3)^2+6^2})^2} dx$$

$$u = x - 3$$

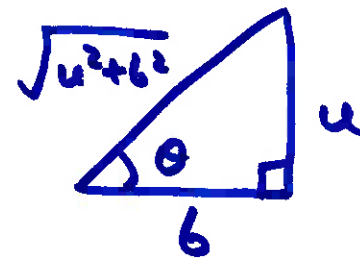
$$du = dx$$

$$\int \frac{1}{(\sqrt{u^2+6^2})^2} du$$

triangle w/ sides $\sqrt{u^2+6^2}$, u , 6

hypotenuse: $\sqrt{u^2+6^2}$

adjacent: 6



$$\text{Let } \tan \theta = \frac{u}{6} \rightarrow u = 6 \tan \theta$$

$$du = 6 \sec^2 \theta d\theta$$

$$\int \frac{1}{\underbrace{(\sqrt{36 \tan^2 \theta + 36})^2}_{36(\tan^2 \theta + 1)} \cdot 6 \sec^2 \theta d\theta} = \int \frac{1}{36 \sec^2 \theta} \cdot 6 \sec^2 \theta d\theta$$

$$= \int \frac{1}{b} d\theta = \frac{1}{b} \theta + C$$

now back to u , back to x

$$\text{from } u = b \tan \theta \rightarrow \theta = \tan^{-1}\left(\frac{u}{b}\right) = \tan^{-1}\left(\frac{x-3}{b}\right)$$

$$= \boxed{\frac{1}{b} \tan^{-1}\left(\frac{x-3}{b}\right) + C}$$

example

$$\int \frac{1}{\sqrt{(x-5)(1-x)}} dx$$

make what's under radical a sum or difference of squares

$$(x-5)(1-x) = -x^2 + 6x - 5$$

$$= - (x^2 - 6x + 5) \quad \text{complete square inside ()}$$

$$= - (x^2 - 6x + 9 + 5 - 9)$$

$$= - [(x^2 - 6x + 9) - 4] = - [(x-3)^2 - 2^2]$$

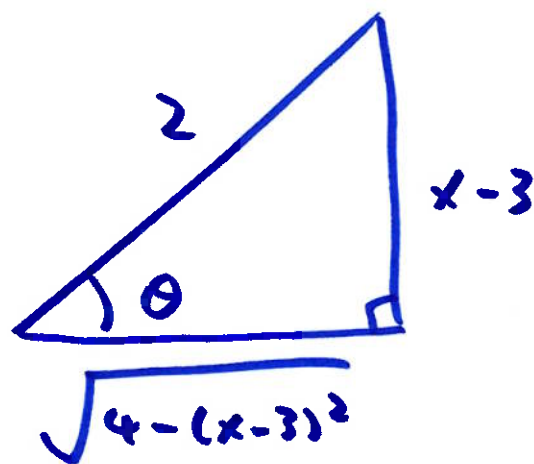
$$= 2^2 - (x-3)^2$$

$$\int \frac{1}{\sqrt{2^2 - (x-3)^2}} dx$$

triangle w/ sides $\sqrt{4-(x-3)^2}$, 2, $x-3$

hypotenuse: 2

adjacent: $\sqrt{4-(x-3)^2}$



$$\sin \theta = \frac{x-3}{2}$$

$$x-3 = 2 \sin \theta$$

$$x = 2 \sin \theta + 3$$

$$dx = 2 \cos \theta d\theta$$

$$\int \frac{1}{\underbrace{\sqrt{4-(x-3)^2}}_{2 \sin \theta}} dx = \int \frac{1}{\underbrace{\sqrt{4-4 \sin^2 \theta}}_{4(1-\sin^2 \theta) \quad 4 \cos^2 \theta}} \cdot 2 \cos \theta d\theta = \int \frac{1}{\sqrt{4 \cos^2 \theta}} \cdot 2 \cos \theta d\theta$$

$$= \int d\theta = \theta + C$$

from $\sin \theta = \frac{x-3}{2}$ we get $\theta = \sin^{-1} \left(\frac{x-3}{2} \right)$

$$= \boxed{\sin^{-1} \left(\frac{x-3}{2} \right) + C}$$