8.5 Partial Fractions Expansions (part 1)

$$\int \frac{x-8}{x^2-7x+10}\,dx$$

options: direct integration no.

substitution (u) no

by parts maybe, but messy

trig subs maybe, but messy

if we could recognize that $\dfrac{x-8}{x^2-7x+10} = \dfrac{2}{x-2} - \dfrac{1}{x-5}$

then $$\int \frac{x-8}{x^2-7x+10}\,dx = \int \left(\frac{2}{x-2} - \frac{1}{x-5}\right)dx$$

$$= \int \frac{2}{x-2}\,dx - \int \frac{1}{x-5}\,dx$$

$$= \boxed{2\ln|x-2| - \ln|x-5| + C}$$

How do we do partial fractions expansion?

Case 1 : Denominator is a product of distinct linear factors

not repeated first order (power is 1)

$$\frac{x-8}{x^2-7x+10} = \frac{x-8}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2}$$

A, B : constants

now find A and B

$$\frac{x-8}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{(x-2)}$$

multiply through by $(x-5)(x-2)$

$$x-8 = A(x-2) + B(x-5)$$

$$= Ax - 2A + Bx - 5B$$

$$x - 8 = (A+B)x + (-2A-5B)$$

$A+B = 1$ — ①

$-2A-5B = -8$ — ②

from ① $\rightarrow$ $B = 1 - A$ sub into ②

$$-2A - 5B = -8$$

$$-2A - 5(1-A) = -8$$

$$-2A - 5 + 5A = -8$$

$$3A = -3 \rightarrow A = -1$$

$$B = 1 - A = 1 - (-1) = 2$$

So,
$$\frac{x-8}{x^2-7x+10} = \frac{A}{x-5} + \frac{B}{x-2} = -\frac{1}{x-5} + \frac{2}{x-2}$$

Same as shown on first page

example

$$\int \frac{1}{x^2-4}\,dx$$

$$\frac{1}{x^2-4} = \underbrace{\frac{1}{(x-2)(x+2)}}_{\substack{\text{product of} \\ \text{distinct linear} \\ \text{factors}}} = \frac{A}{x-2} + \frac{B}{x+2}$$

multiply by $(x-2)(x+2)$

$$1 = A(x+2) + B(x-2)$$

$$= Ax + 2A + Bx - 2B$$

$$0x + 1 = \underline{\underline{(A+B)}}\,x + \underline{\underline{(2A-2B)}}$$

$A+B=0$ — ①

$2A-2B=1$ — ②

another way to solve the system

multiply ① by 2

$$2A+2B=0 \quad -③$$
$$2A-2B=1 \quad -②$$

add ③ and ② $\rightarrow$ $4A=1 \rightarrow A=\frac{1}{4}$, $B=-\frac{1}{4}$ (from any eq., for example ①)

$$\int \frac{1}{x^2-4}\,dx = \int\left(\frac{A}{x-2}+\frac{B}{x+2}\right)dx$$

$$= \int \frac{1}{4}\frac{1}{x-2}\,dx + \int -\frac{1}{4}\frac{1}{x+2}\,dx$$

$$= \boxed{\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C}$$

Case 2: Denominator is a product of linear factors and some of which are repeated

$$\frac{10}{5x^2-2x^3} = \frac{10}{(x^2)(5-2x)} = \frac{10}{\underbrace{(x)(x)}_{\text{repeated linear factor}}(5-2x)}$$

for repeated factors

$$\frac{10}{(x)(x)(5-2x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-2x}$$

each time the factor appears, bump power up by 1

multiply by $(x)(x)(5-2x)$

$$10 = A(x)(5-2x) + B(5-2x) + C(x)(x)$$

another way to find constants: pick x such that as many of them as possible disappear pick $x = 0$

$$10 = B(5-0) \rightarrow 10 = 5B \rightarrow \underline{B = 2}$$

pick $x = 5/2$ (such that $5-2x=0$)

$$10 = C\left(\frac{5}{2}\right)\left(\frac{5}{2}\right) \rightarrow 10 = \frac{25}{4}C \rightarrow \underline{C = \frac{8}{5}}$$

for the last constant, choose any convenient x and use the constants we already know

pick $x = 1$

$$10 = 3A + 3\overset{2}{B} + \overset{8/5}{C}$$

$$= 3A + 6 + \frac{8}{5} \rightarrow A = 4/5$$

$$\frac{10}{5x^2 - 2x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-2x} = \frac{4}{5}\cdot\frac{1}{x} + \frac{2}{x^2} + \frac{8}{5}\,\frac{1}{5-2x}$$

$$\int \frac{10}{5x^2-2x^3}\,dx = \int \left(\frac{4/5}{x} + \frac{2}{x^2} + \frac{8/5}{5-2x} \right) dx$$

$$= \cdots = \boxed{\frac{4}{5}\ln|x| - \frac{2}{x} - \cancel{\frac{8}{5}}\,\frac{4}{5}\ln|5-2x| + D}$$

if we have

$$\frac{1}{\underbrace{(x)(x)}_{\text{twice}}\underbrace{(x-1)(x-1)(x-1)}_{\text{thrice}}(x+10)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{F}{x+10}$$

Important : for the expansion to work, the numerator of the original expression MUST have a lower degree the denominator

$$\frac{x-8}{x^2-7x+10}$$

first degree (1)

second degree (2)

is ok.

$$\frac{-2x^3+5x^2+10}{-2x^3+5x^2}$$

3rd

3rd

is not ok

BEFORE expansion, we need to do

$$\frac{-2x^3+5x^2}{-2x^3+5x^2} + \frac{10}{-2x^3+5x^2} = 1 + \frac{10}{-2x^3+5x^2}$$

can expand as we did w/ previous examples