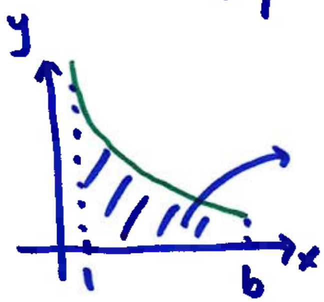
8.9 Improper Integrals

$f(x) = \frac{1}{x}$

we know $\int_1^b \frac{1}{x}\,dx$ is the area under $\frac{1}{x}$ from $x=1$ to $x=b$



$$\int_1^b \frac{1}{x}\,dx = \ln|x|\Big|_1^b = \ln b - \ln 1 = \ln b$$

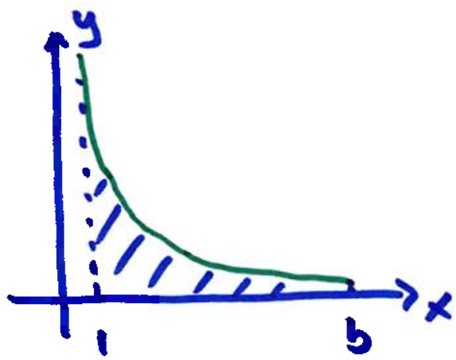
what if $b \to \infty$

$$\underbrace{\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx} = \lim_{b\to\infty} \ln b = \infty$$ → the integral is <u>unbounded</u> as $b \to \infty$

$$\int_1^\infty \frac{1}{x}\,dx$$ this is a type of <u>improper integrals</u> → at least one integration limit is ∞ or $-\infty$

let's try $f(x) = \frac{1}{x^2}$

$$\int_1^b \frac{1}{x^2}\,dx$$



$$\int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b = -\frac{1}{x}\Big|_1^b = -\frac{1}{b}+1$$

as $b \to \infty$, $\displaystyle\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$

the area under $\frac{1}{x^2}$ is bounded even as $b \to \infty$

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

if the improper integral goes to ∞ or $-\infty$, we say the integral <u>diverges</u> or is <u>divergent</u> (e.g. $\int_1^\infty \frac{1}{x}\,dx$)

if the improper integral resultes in a number, we say the integral <u>converges</u> or is <u>convergent</u> (e.g. $\int_1^\infty \frac{1}{x^2}\,dx$)

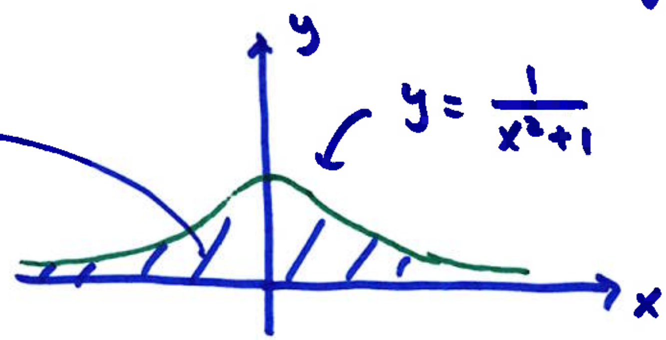
it turns out $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$

diverges if $p \le 1$

the difference is how fast $\frac{1}{x^p}$ decreases

the improper integral can have both integration limits being ∞ and $-\infty$

<u>example</u> $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$



$$= \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+1}\,dx + \lim_{a\to\infty} \int_{0}^{a} \frac{1}{x^2+1}\,dx$$

$$= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_{a}^{0} + \lim_{a\to\infty} \tan^{-1}(x)\Big|_{0}^{a}$$

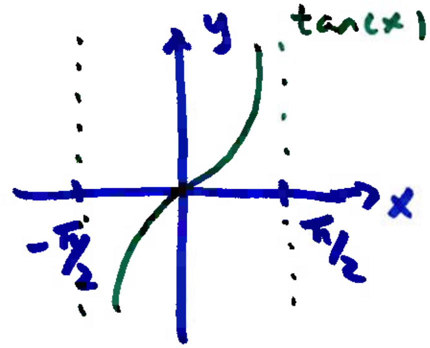
$$= \lim_{a\to-\infty}\left(\underbrace{\tan^{-1}(0)}_{0} - \tan^{-1}(a)\right) + \lim_{a\to\infty}\left(\tan^{-1}(a) - \underbrace{\tan^{-1}(0)}_{0}\right)$$

$$= \lim_{a\to-\infty}\left(-\tan^{-1}(a)\right) + \lim_{a\to\infty}\left(\tan^{-1}(a)\right)$$

$\tan^{-1}(\infty) = ?$

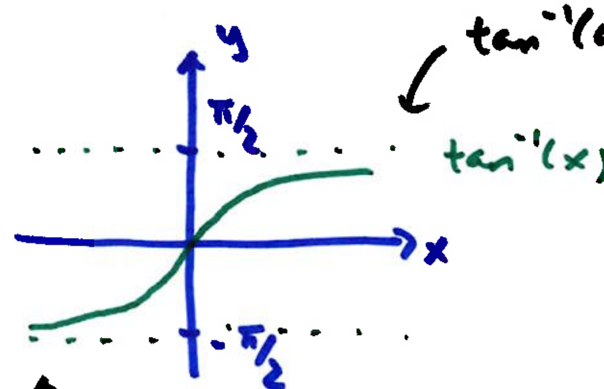
$\tan^{-1}(-\infty) = ?$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$



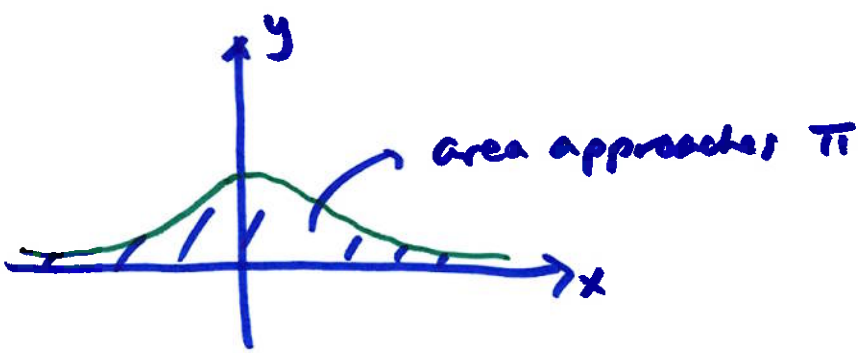
$$= \boxed{\pi}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi \quad \text{(convergent)}$$



$\tan^{-1}(\infty) = \frac{\pi}{2}$

$\tan^{-1}(-\infty) = -\pi/2$



another type of improper integrals: integration limits are finite but the integrand becomes undefined at some point down on the interval

$\int_a^b f(x)\,dx$ also improper if $f(x) \to \infty$ or $-\infty$ somewhere on $a \le x \le b$

<u>example</u> $$\int_{-2}^{3} \frac{1}{x^4}\,dx$$

note $\frac{1}{x^4} \to \infty$ as $x \to 0$ which is in $-2 \le x \le 3$

so this is an improper integral

we want to stay away from $x = 0$ (where $\frac{1}{x^4} \to \infty$ (or $-\infty$))

$$\int_{-2}^{0} \frac{1}{x^4}\,dx + \int_{0}^{3} \frac{1}{x^4}\,dx$$

$$= \lim_{b \to 0^-} \int_{-2}^{b} \frac{1}{x^4}\,dx + \lim_{a \to 0^+} \int_{a}^{3} \frac{1}{x^4}\,dx$$

$$= \lim_{b\to 0^-}\left(-\frac{1}{3x^3}\right)\Big|_{-2}^{b} + \lim_{a\to 0^+}\left(-\frac{1}{3x^3}\right)\Big|_{a}^{3}$$

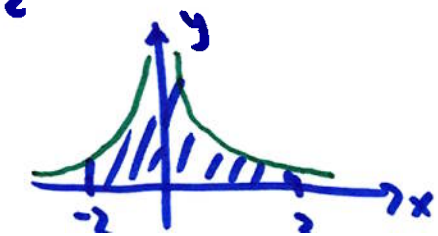
$$= \underbrace{\lim_{b\to 0^-}}_{\substack{\text{b is a}\\ \text{small}\\ \text{neg. \#}}}\left(-\underbrace{\frac{1}{3b^3}}_{\substack{\text{large}\\ \text{pos. \#}}} - \frac{1}{24}\right) + \underbrace{\lim_{a\to 0^+}}_{\substack{\text{a is a}\\ \text{small}\\ \text{pos. \#}}}\left(-\frac{1}{81} + \underbrace{\frac{1}{3a^3}}_{\substack{\text{large}\\ \text{pos. \#}}}\right)$$

$$= \infty - \frac{1}{24} - \frac{1}{81} + \infty = \boxed{\infty} \quad \text{(divergent)}$$

improper integrals of this type can be easily missed and wrong results will result

$$\int_{-2}^{3} \frac{1}{x^4}\,dx$$ pretend we didn't realize this is improper

$$= -\frac{1}{3x^3}\Big|_{-2}^{3} = -\frac{1}{81} - \frac{1}{24} = -\frac{35}{648}$$ wrong! completely meaningless



we can compare integrals to (sometimes) quickly determine if improper integral will converge

for example, we found that $\int_1^\infty \frac{1}{x^2} = 1$ (converges)

Since $0 \leq \frac{1}{x^2+1} \leq \frac{1}{x^2}$ because $\frac{1}{x^2+1}$ has larger denominator

$$0 \leq \underbrace{\int_1^\infty \frac{1}{x^2+1}\,dx}_{\text{must also converge}} \leq \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{convergent so is finite}}$$

Similarly, $\int_1^\infty \frac{1}{x}\,dx$ diverges

and $\frac{1}{x-\frac{1}{2}} \geq \frac{1}{x}$ since $\frac{1}{x-\frac{1}{2}}$ has small denominator

must be a bigger ∞

$$\underbrace{\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx}_{} \geq \underbrace{\int_1^\infty \frac{1}{x}\,dx}_{\infty}$$

so $\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx$ diverges