10.3 Infinite Series

NOT on exam 2

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

a_k : k^{th} term, usually formula is given

many kinds of series to study

today: geometric series

telescoping series

geometric series: common ratio between the terms

for example, $\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots$ ratio: $r = \frac{1}{3}$

we can write a geometric series in this form:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{k=0}^{\infty} ar^k$$

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots = \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)$$

$$= \underbrace{\tfrac{1}{4}}_{a} + \underbrace{\tfrac{1}{4}}_{a}\underbrace{\left(\tfrac{1}{3}\right)}_{r} + \underbrace{\tfrac{1}{4}}_{a}\underbrace{\left(\tfrac{1}{3}\right)^2}_{r} + \tfrac{1}{4}\underbrace{\left(\tfrac{1}{3}\right)^3}_{r} + \cdots = \boxed{\sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{1}{3}\right)^k}$$

the starting k value doesn't really matter

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

notice $\displaystyle\sum_{k=1}^{\infty} ar^{k-1}$ is the same series

$$= \underset{k=1}{ar^{1-1}} + \underset{k=2}{ar^{2-1}} + \underset{k=3}{ar^{3-1}} + \dots$$

$$= ar^0 + ar + ar^2 + \dots = a + ar + ar^2 + ar^3 + \dots$$

Same

the partial sums of a geometric series

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \frac{1}{324} + \frac{1}{972} + \dots$$

1st partial sum

$$S_1 = \frac{1}{4}$$

$$S_2 = \frac{1}{4} + \frac{1}{12}$$

$$S_3 = \frac{1}{4} + \frac{1}{12} + \frac{1}{36}$$

$$\vdots$$

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

$S_1 = a = ar^0$

$S_2 = a + ar^1$

$S_3 = a + ar + ar^2$

$S_4 = a + ar + ar^2 + ar^3$

pattern: n^{th} partial sum ends w/ $n-1$ power of r

$\vdots$

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ — ①

let's find a formula for S_n w/o adding n terms

multiply by r

$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$ — ②

① − ②

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a - ar^n \longrightarrow \boxed{S_n = \dfrac{a - ar^n}{1-r}}$

$$\frac{1}{4} + \frac{1}{36} + \frac{1}{108} + \cdots = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{1}{3}\right)^k \rightarrow a = \frac{1}{4}, \quad r = \frac{1}{3}$$

7^{th} partial sum (w/o adding 7 terms)

$$S_7 = \frac{a - ar^7}{1-r} = \frac{\frac{1}{4} - \frac{1}{4}\left(\frac{1}{3}\right)^7}{1 - \frac{1}{3}} = \cdots = \boxed{\frac{1093}{2916}}$$

bigger question: does $S_n \to$ finite value as $n \to \infty$

in other words, does $\sum_{k=0}^{\infty} ar^k$ converge? $\lim_{n\to\infty} S_n = L$?

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} \underbrace{(1-r^n)}$$

limit exists only if r^n ~~$\to 0$~~ exists as $n \to \infty$

and that happens if $|r| < 1$ or $-1 < r < 1$

so, for $\sum_{k=0}^{\infty} ar^k$, it converges if $|r| < 1$

and its sum is $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-\overbrace{r^n}^{\text{goes to } 0})}{1-r} = \boxed{\frac{a}{1-r}}$

first term

common ratio

try on $\underbrace{\frac{1}{4}}_{a} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots = \sum_{k=0}^{\infty} \underbrace{\frac{1}{4}}_{a}\left(\underbrace{\frac{1}{3}}_{r}\right)^k$

$$= \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{3}} = \frac{\frac{1}{4}}{\frac{2}{3}} = \boxed{\frac{3}{8}}$$

$\underbrace{1}_{a} - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \sum_{k=0}^{\infty} \left(\underbrace{-\frac{1}{2}}_{r}\right)^k = \frac{a}{1-r} = \frac{1}{1--\frac{1}{2}} = \frac{2}{3}$

Telescoping Series

<u>example</u> $$\sum_{k=1}^{\infty} \frac{1}{(k+11)(k+12)}$$

$$= \underbrace{\frac{1}{12\cdot 13}}_{k=1} + \underbrace{\frac{1}{13\cdot 14}}_{k=2} + \frac{1}{14\cdot 15} + \frac{1}{15\cdot 16} + \cdots = ?$$ not clear where this is going

partial fraction: $$\frac{1}{(k+11)(k+12)} \overset{= \frac{A}{k+11} + \frac{B}{k+12}}{=} \frac{1}{k+11} - \frac{1}{k+12}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k+11} - \frac{1}{k+12} \right)$$

$$= \underbrace{\left(\frac{1}{12} - \cancel{\frac{1}{13}}\right)}_{k=1} + \underbrace{\left(\cancel{\frac{1}{13}} - \cancel{\frac{1}{14}}\right)}_{k=2} + \left(\cancel{\frac{1}{14}} - \frac{1}{15}\right) + \cdots$$

$S_3 = \frac{1}{12} - \frac{1}{15}$ only first and last survive

$S_4 = \frac{1}{12} - \frac{1}{16}$

$S_5 = \frac{1}{12} - \frac{1}{17}$

now generalize for S_n

$$S_n = \frac{1}{12} - \frac{1}{12+n}$$

does this series converge? yes, because $\lim_{n\to\infty} S_n$ exists

what does it converge to? $\frac{1}{12}$

if we keep adding terms, the sum approaches $\frac{1}{12}$

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the starting K does NOT affect convergence

if a series converges, changing starting K ~~to~~ results in a convergent series

for example, $\sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K$ converges because $|r| < 1$

$$= \underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)}_{\sum_{K=0}^{3} \left(\frac{1}{2}\right)^K} + \underbrace{\left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots\right)}_{\sum_{K=4}^{\infty} \left(\frac{1}{2}\right)^K} = 2 \qquad = \frac{1}{1-\frac{1}{2}} = 2$$

finit series ALWAYS converges

must also converge because it's a part of a convergent series

EXAM REVIEW
MON 10/16
5:30 - 7:30 pm
University Church 114