10.4 The Divergence and Integral Tests

How do we know if an infinite series $\sum_{k=1}^{\infty} a_k$ converges?

we will see several "tests" to test if a converges.

last time: geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots$

converges if $|r| < 1$

what about $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^2}$

Let's look at the convergence question from the opposite point of view

→ if a series converges, what must happen?

$\sum_{k=1}^{\infty} a_k$

$S_1 = a_1$ first partial sum

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_{k-1} = a_1 + a_2 + a_3 + \ldots + a_{k-1}$

$S_k = a_1 + a_2 + a_3 + \ldots + a_{k-1} + a_k$

convergence: $\lim_{k \to \infty} S_k = L$ (a finite number)

$$\lim_{k \to \infty} S_{k-1} = L$$

from last page, note $S_2 - S_1 = a_2$

$$S_3 - S_2 = a_3$$

$$S_k - S_{k-1} = a_k$$

then $\lim_{k \to \infty} (S_k - S_{k-1}) = \lim_{k \to \infty} a_k$

$$\underbrace{\lim_{k \to \infty} S_k}_{L} - \underbrace{\lim_{k \to \infty} S_{k-1}}_{L} = \boxed{\lim_{k \to \infty} a_k = 0}$$

This is the <u>Divergence Test</u>

<u>if</u> $\sum_{k=1}^{\infty} a_k$ converges, <u>then</u> $\lim_{k \to \infty} a_k = 0$

this is a one way if

the converse is not necessarily true

for example, $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$ is a geometric series with $|r| < 1$ $\left(r = \frac{1}{3}\right)$

so this converges

notice $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \left(\frac{1}{3}\right)^k = \lim_{k \to \infty} \frac{1}{3^k} = 0$

but $\sum_{k=0}^{\infty} \cos(k\pi) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

clearly diverges, note $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \cos(k\pi)$ DNE (not zero)

but, just because $\lim_{k \to \infty} a_k = 0$, it does **NOT** necessarily mean $\sum_{k=1}^{\infty} a_k$ converges.

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$ "Harmonic Series"

clearly, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$

but does it converge?

Let's look at some partial sums

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = 1.5$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = 1.833$

$S_4 = 2.0833$

$\vdots$

$S_{20} = 3.5977$

$\vdots$

$S_{50} = 4.4992$

$\vdots$

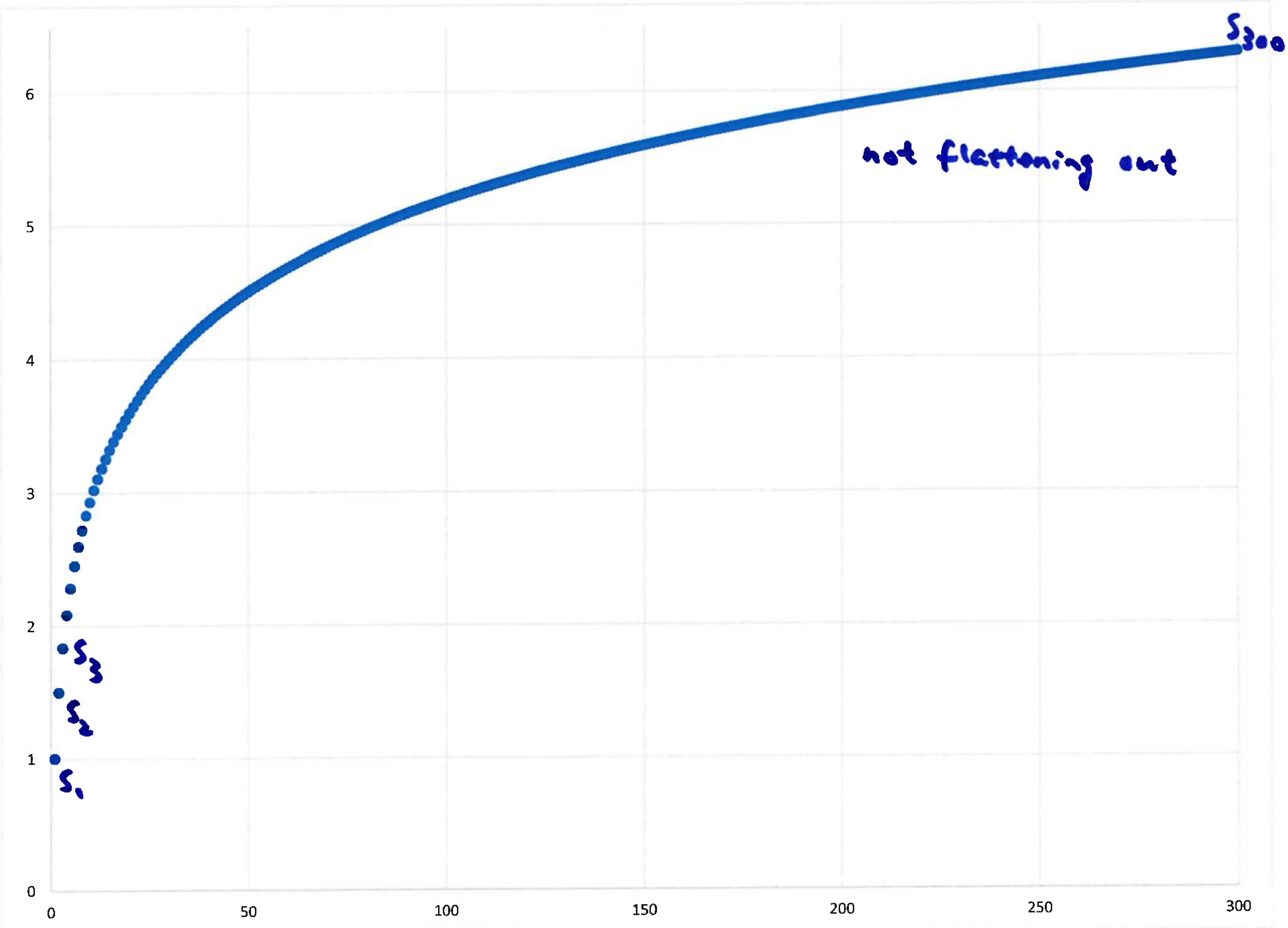
$S_{100} = 5.1874$

$\vdots$

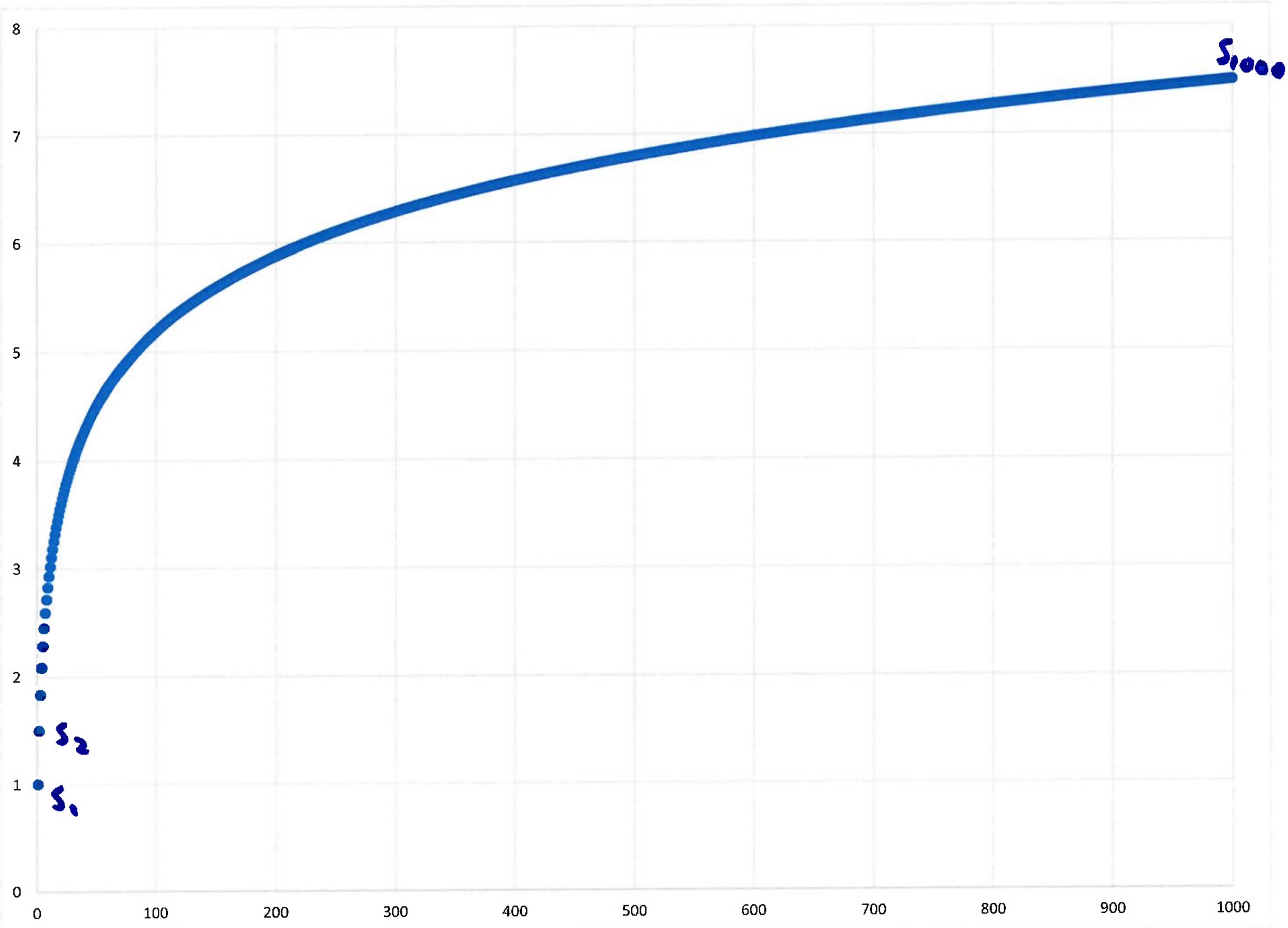
$S_{500} = 6.7928$

there is no sign of S_k settling down

$$\sum_{k=1}^{\infty} \frac{1}{k}$$



$$\sum_{k=1}^{\infty} \frac{1}{k}$$



the Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k}$ has $\lim_{k\to\infty} a_k = 0$ but does NOT converge

So, again, just because $\lim_{k\to\infty} a_k = 0$ it does NOT mean $\sum_{k=1}^{\infty} a_k$ converges

BUT, if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

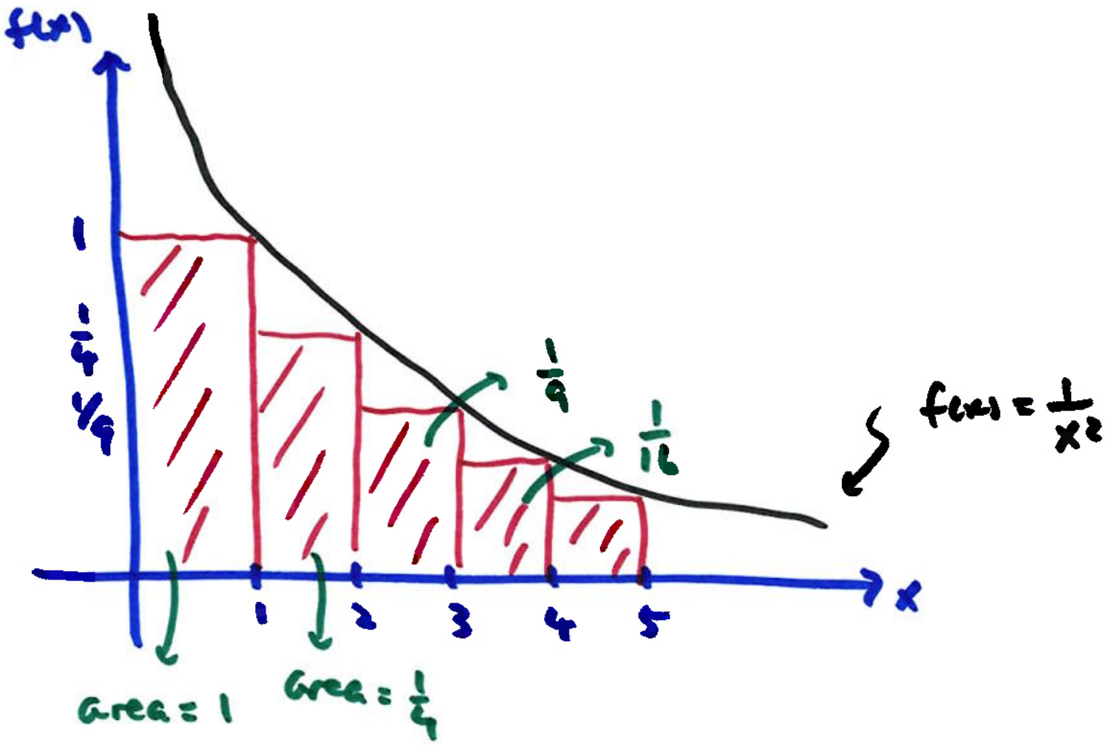
So, if we know $\cancel{\sum_{k=}}$ $\lim_{k\to\infty} a_k = 0$, how do we make sure that the series $\sum_{k=1}^{\infty} a_k$ converges?

today, we will see the <u>Integral Test</u>

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots}$$

treat as points on graph of $f(x) = \frac{1}{x^2}$ at $x = 1, 2, 3, 4, \ldots$

$\lim_{k\to\infty} \frac{1}{k^2} = 0$, passes the Divergence Test but not guaranteed to converge graph



look at $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

as a Riemann sum approximating the integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

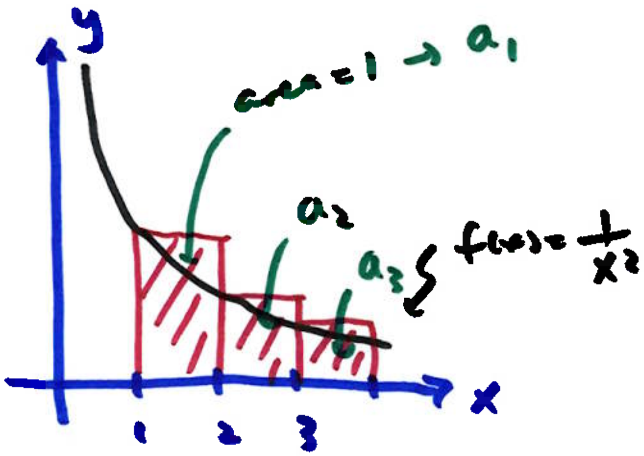
notice the boxes are below the curve, so the approx. is an underestimate

So, $\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} \le 1 + \int_1^\infty \frac{1}{x^2}\,dx}$

first box

we will revisit this

now we look at $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ as a Right Riemann Sum



this estimate is an over estimate

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} \geq \int_1^{\infty} \frac{1}{x^2}\,dx}$$

Combine the two black boxes

$$\int_1^{\infty} \frac{1}{x^2}\,dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

so, if $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, then $\sum_{k=1}^{\infty} \frac{1}{k^2}$ also converges

this is the Integral Test.

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \underbrace{a(k)}_{\hookrightarrow f(x)}$$

$\int_1^{\infty} f(x)\,dx$ if conv. then series conv.

So, does $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge?

$\rightarrow$ does $\int_1^{\infty} \frac{1}{K^2}\,dx$ converge? yes, in fact $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$

So, that means

$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{K^p}}_{\text{p-series}} \quad \underbrace{\text{converges if } p > 1}_{\text{"p-series test"}}$$

for example, $\sum_{k=1}^{\infty} \frac{1}{K^7}$ converges because $p = 7 > 1$

and $\sum_{k=1}^{\infty} \frac{1}{K}$ diverges because $p = 1$

the divergence test ($\lim_{k\to\infty} a_k = 0$?) can save us time.

if $\lim_{k\to\infty} a_k \neq 0$, no need to use Integral (or any other) test.

if $\lim_{k\to\infty} a_k = 0$, then we investigate more.

the Integral Test can establish convergence, but does not give us the sum (where the series converges to)

a consequence of the Integral Test is the p-series Test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges if } p > 1$$

the starting k does NOT affect convergence

if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1337}^{\infty} a_k$ also converges

if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1337}^{\infty} a_k$ still diverges.