10.5 Comparison Tests

last time: Divergence Test : $\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{k\to\infty} a_k \neq 0$

but, just because $\lim_{k\to\infty} a_k = 0$ it does NOT necessarily mean $\sum_{k=1}^{\infty} a_k$ converges.

Integral Test: $\sum_{k=1}^{\infty} a_k$ converges if $\int_1^{\infty} a(x)\,dx$ converges

~~fcn~~: function based on a_k

p-series Test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$

Comparison: compare an unknown series to one that we know

$$\cancel{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}} + \cdots = \sum$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

converges

geo. series $r = \frac{1}{2}$ (<1)

what about

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \cdots = ?$$

this is not a geo series or a p-series

notice

$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots \le 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ $\left(\text{sum} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2 \right)$

because each term on the series on the left $\le$ that of the right series

so, $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots \le 2$ $\rightarrow$ so this series must also converge

<u>example</u> $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+5}$

always check if $\displaystyle\lim_{k\to\infty} a_k = 0$ (Div. Test)

does it pass the Div. Test? Yes, test more.

compare to what?

$\rightarrow$ think about what the series looks like if k is large

$\frac{1}{k^3+5}$ as $k \to \infty$ $\frac{1}{k^3+5} \approx \frac{1}{k^3}$ so, compare to $\sum \frac{1}{k^3}$

does $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converge? (p-series, $p = 3 > 1$)

$$\frac{1}{k^3+5} \leq \frac{1}{k^3} \quad \text{for } k = 1, 2, 3, \ldots$$

so, $\sum_{k=1}^{\infty} \frac{1}{k^3+5} \leq \sum_{k=1}^{\infty} \frac{1}{k^3} = S$ (it has a finite sum because it converges)

therefore, $\sum_{k=1}^{\infty} \frac{1}{k^3+5} \leq S$ therefore converges.

<u>example</u> $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$

passes the Div. Test since $\lim_{k \to \infty} \frac{k+1}{k^2} = 0$

compare to what happens when k is large

$\frac{k+1}{k^2} \approx \frac{k}{k^2} = \frac{1}{k}$ so compare to $\sum_{k=1}^{\infty} \frac{1}{k}$ (divergent) for $k = 1, 2, 3, 4, 5, \ldots$

$$\frac{k+1}{k^2} \geq \frac{1}{k} \quad \text{because} \quad \frac{1}{k} = \frac{k}{k^2}$$

therefore, $$\sum_{k=1}^{\infty} \frac{k+1}{k^2} \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty \quad \text{(diverges)}$$

$$\sum_{k=1}^{\infty} \frac{k+1}{k^2} = \infty \quad \text{(bigger } \infty) \quad \text{so diverges}$$

what about $\sum_{k=1}^{\infty} \frac{k-1}{k^2}$?

$$\frac{k-1}{k^2} \leq \frac{1}{k}$$

so, $$\sum_{k=1}^{\infty} \frac{k-1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

$$\sum_{k=1}^{\infty} \frac{k-1}{k^2} = \infty \quad \text{(a smaller } \infty \text{ OR a finite number)}$$

this comparison does NOT give a conclusive answer to convergence question if the terms of the unknown series are $\boxed{\leq}$ those of a $\boxed{\text{convergent}}$ series, then the unknown series $\boxed{\text{converges}}$

if the terms of the unknown series are $\boxed{\geq}$ those of a $\boxed{\text{divergent}}$ series → $\boxed{\text{diverges}}$

if the terms of the unknown series are $\boxed{\leq}$ those of a $\boxed{\text{divergent}}$ series → $\boxed{?}$ inconclusive (choose a different test)

(same if terms ≥ convergent series)

assumption: terms are positive

a variation is the **Limit Comparison Test**

$\sum_{k=1}^{\infty} b_k$ is a **known** series

$\sum_{k=1}^{\infty} a_k$ is the unknown series

if $\lim_{k\to\infty} \frac{a_k}{b_k} = c$ ($c>0$ but finite)

then BOTH $\sum_{k=1}^{\infty} a_k$ AND $\sum_{k=1}^{\infty} b_k$ converge or BOTH diverge

because this means as $k \to \infty$, $a_k \approx c\, b_k$

so $\sum a_k \approx \sum c\, b_k$

in other words, the tails look a like.

<u>example</u> $\sum_{k=1}^{\infty} \left(\frac{k}{2k+3} \right)^k$

does it pass the Div. Test? $\lim_{k \to \infty} \left(\frac{k}{2k+3} \right)^k = 0$?

as $k \to \infty$, $\frac{k}{2k+3} \approx \frac{k}{2k} \approx \frac{1}{2}$

so $k \to \infty$, $\left(\frac{k}{2k+3} \right)^k \approx \left(\frac{1}{2} \right)^k = 0$

this is suggesting we compare to $\sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k \to$ converges

let $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k$

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{k}{2k+3} \right)^k$

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\left(\frac{k}{2k+3} \right)^k}{\left(\frac{1}{2} \right)^k} = \lim_{k \to \infty} \left(\frac{\frac{k}{2k+3}}{\frac{1}{2}} \right)^k$$

$$= \lim_{k\to\infty} \left(\frac{k}{2k+3} \cdot \frac{2}{1}\right)^k = \lim_{k\to\infty} \left(\frac{2k}{2k+3}\right)^k = \frac{1}{e^{3/2}} > 0 \text{ (and not } \infty)$$

So, BOTH $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ AND $\sum_{k=1}^{\infty} \left(\frac{k}{2k+3}\right)^k$ converge or diverge

Since $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ converges (geo. series $r = \frac{1}{2}$)

$\sum_{k=1}^{\infty} \left(\frac{k}{2k+3}\right)^k$ converges too.