10.6 Alternating Series

Series w/ alternating signs

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$ Alternating Harmonic Series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$ Leibnitz Series

$$= \frac{\pi}{4}$$ (converges to $\frac{\pi}{4}$)

$(-1)^k$ or variations $\rightarrow$ alternating signs

power of (-1) shifted by 2

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

shifting power by 2 while keeping the same starting k does not change the series

always non negative

general form: $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ $\qquad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

a_k

two things need to happen for an alternating series to converge

1) $a_{k+1} \leq a_k$ after some k

→ magnitude does not get bigger

2) $\lim_{k \to \infty} a_k = 0$

→ the divergence test

The Alternating Series Test

look at the alternating Harmonic series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

clearly, $\lim_{k \to \infty} \frac{1}{k} = 0$ (passes the divergence test)

and $\frac{1}{k}$ clearly decreases as k increases

$$\frac{1}{k+1} \leq \frac{1}{k} \quad \text{for any } k$$

so, the Alt. Harmonic Series <u>converges</u> (even though the "regular" Harmonic does not) what does $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ converge to ?

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \overbrace{\underbrace{1}_{S_1} - \frac{1}{2}}^{S_2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

partial sums

$S_1 = 1$

$S_2 = 1 - \frac{1}{2} = \frac{1}{2}$

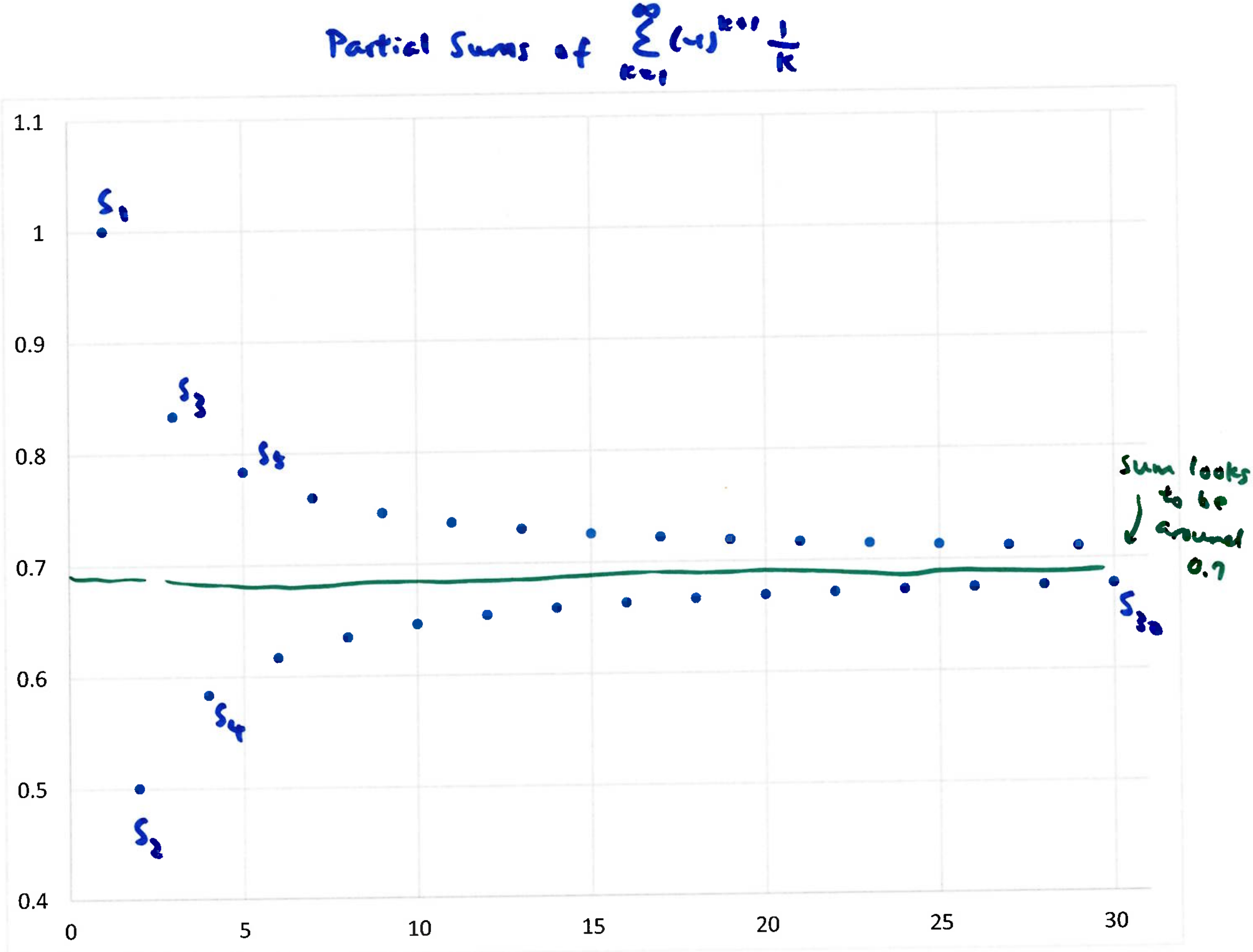
$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = 0.8333$

$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = 0.5833$

$S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.7833$

what we add back $<$ what we took away

should settle down


Partial Sums of $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
5
10
15
20
25
30
S_1
S_2
S_3
S_4
S_5
S_{30}
Sum looks to be around 0.7

<u>example</u> Does $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^6+9}$ converge?

$$= \frac{1}{9} - \frac{1}{10} + \frac{1}{73} - \dots$$

to converge: $\lim_{k\to\infty} \frac{1}{k^6+9} = 0$? yes

is $\frac{1}{k^6+9}$ nonincreasing? yes

to be sure: $\frac{d}{dk}\left(\frac{1}{k^6+9}\right) \leq 0$

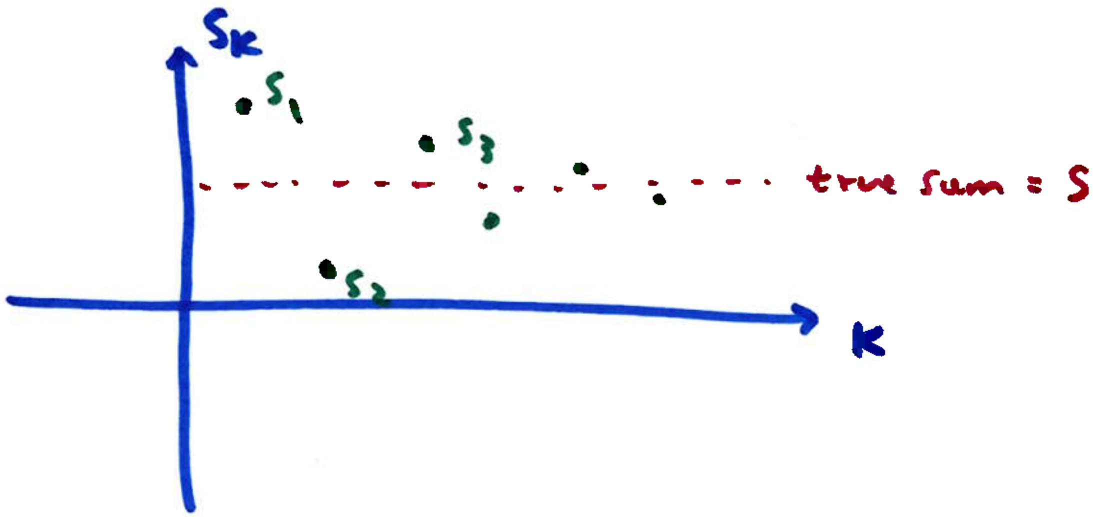
$$\frac{\overbrace{-6k^5}^{\text{neg. for } k>0}}{\underbrace{(k^6+9)^2}_{\text{pos. (squared)}}} < 0$$

so a_k is always decreasing

so, $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^6+9}$ converges

Estimating sum of an alternating series

if $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges, then



notice the true sum S is always between two consecutive partial sums

$$S_k \leq S \leq S_{k+1}$$

or

$$S_{k+1} \leq S \leq S_k$$

$$S_k \le S \le S_{k+1}$$

subtract S_k

$$0 \le \underbrace{|S - S_k|}_{\text{how far true sum is from a partial sum}} \le \underbrace{|S_{k+1} - S_k|}_{\text{how far are two partial sums apart}}$$

$|S_{k+1} - S_k| = ?$

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

$$\left.\begin{array}{l} S_1 = a_1 \\ S_2 = a_1 - a_2 \end{array}\right\} |S_2 - S_1| = |a_2|$$

$$\left.\begin{array}{l} S_2 = a_1 - a_2 \\ S_3 = a_1 - a_2 + a_3 \end{array}\right\} |S_3 - S_2| = |a_3|$$

$\vdots$

$$|S_{k+1} - S_k| = |a_{k+1}|$$

So, $\boxed{0 \le |S - S_k| \le |a_{k+1}|}$

the partial sum S_k is no more than the magnitude of the next term away from the true sum example

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

$$= \underbrace{\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}}_{S_4 = \frac{7}{12}} + \underbrace{\boxed{\frac{1}{5}}}_{a_5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

how close is S_4 to the true sum?

$$0 \le |S - S_4| \le |a_5|$$

$$0 \le \left|S - \frac{7}{12}\right| \le \frac{1}{5}$$

so, the true sum of the series is no more than $\frac{1}{5}$ away from $\frac{7}{12}$

$$\frac{7}{12} - \frac{1}{5} \le S \le \frac{7}{12} + \frac{1}{5}$$

~~if~~ if $\sum_{k=1}^{\infty} |a_k|$ converges then we say $\sum_{k=1}^{\infty} a_k$ <u>converges absolutely</u>

for example, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ is <u>absolutely convergent</u>

because $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

if $\sum_{k=1}^{\infty} |a_k|$ diverges BUT $\sum_{k=1}^{\infty} a_k$ converges

$\rightarrow \sum_{k=1}^{\infty} a_k$ <u>converges conditionally</u>

for example, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges but

$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges

so, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is <u>conditionally convergent</u>