11.1 Approximating Functions with Polynomials

NOT on exam 3

Power Series : $\sum_{k=0}^{\infty} c_k (x-a)^k$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

a : center of power series

c_k : coefficients of the k^{th} order term

the power series we will investigate is the Taylor Series

idea: write a power series that behaves like a function $f(x)$ of our choice

Taylor series matches the function value and all derivatives at $x=a$

so near $x=a$, Taylor series acts like the real $f(x)$

Taylor series of $f(x)$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

match function value at $x=a$

$$f(a) = c_0 + c_1\cancel{(a-a)} + c_2\cancel{(a-a)^2} + c_3\cancel{(a-a)^3} + \dots \rightarrow \boxed{f(a) = c_0} = 0!\,c_0$$

now match derivatives at $x=a$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

$$f'(a) = c_1 \rightarrow \boxed{f'(a) = c_1} = 1!\,c_1$$

$$f''(x) = 2c_2 + 3\cdot 2\,c_3(x-a) + 4\cdot 3\,c_4(x-a)^2 + \dots$$

$$f''(a) = 2c_2 \rightarrow \boxed{f''(a) = 2c_2} = 2!\,c_2$$

$$f'''(x) = 3\cdot 2\,c_3 + 4\cdot 3\cdot 2\,c_4(x-a) + \dots$$

$$f'''(a) = 3\cdot 2\,c_3 \rightarrow \boxed{f'''(a) = 3\cdot 2\,c_3} = 3!\,c_3$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2\,c_4 + \dots$$

$$f^{(4)}(a) = 4\cdot 3\cdot 2\,c_4 \rightarrow \boxed{f^{(4)}(a) = 4\cdot 3\cdot 2\,c_4} = 4!\,c_4$$

generalize: $f^{(k)}(a) = k!\,c_k \rightarrow \boxed{c_k = \dfrac{f^{(k)}(a)}{k!}}$

Taylor series of $f(x)$ at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

if we stop at $k=1$ $\rightarrow$ Linear approximation

$$f(x) = f(a) + f'(a)(x-a)$$

think of Taylor series at an extension of linear approx.
more shape features added each k

example Find the Taylor series of $f(x) = e^x$ at $x = a$ $a = 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$f(x) = e^x$ $a = 0$

$f(0) = e^0 = 1$

$f'(x) = e^x$

$f'(0) = e^0 = 1$

$f''(x) = e^x$

$f''(0) = 1$

$\vdots$

$f^{(k)}(0) = 1$

So we get

$$f(x) = 1 + 1\cdot(x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \approx e^x \text{ near } x=0$$

the more terms we include, the better the approx.

* $k \to \infty \Rightarrow$ series converges to e^x

If we cut off after k, we get the kth-order Taylor Polynomial (P_k)

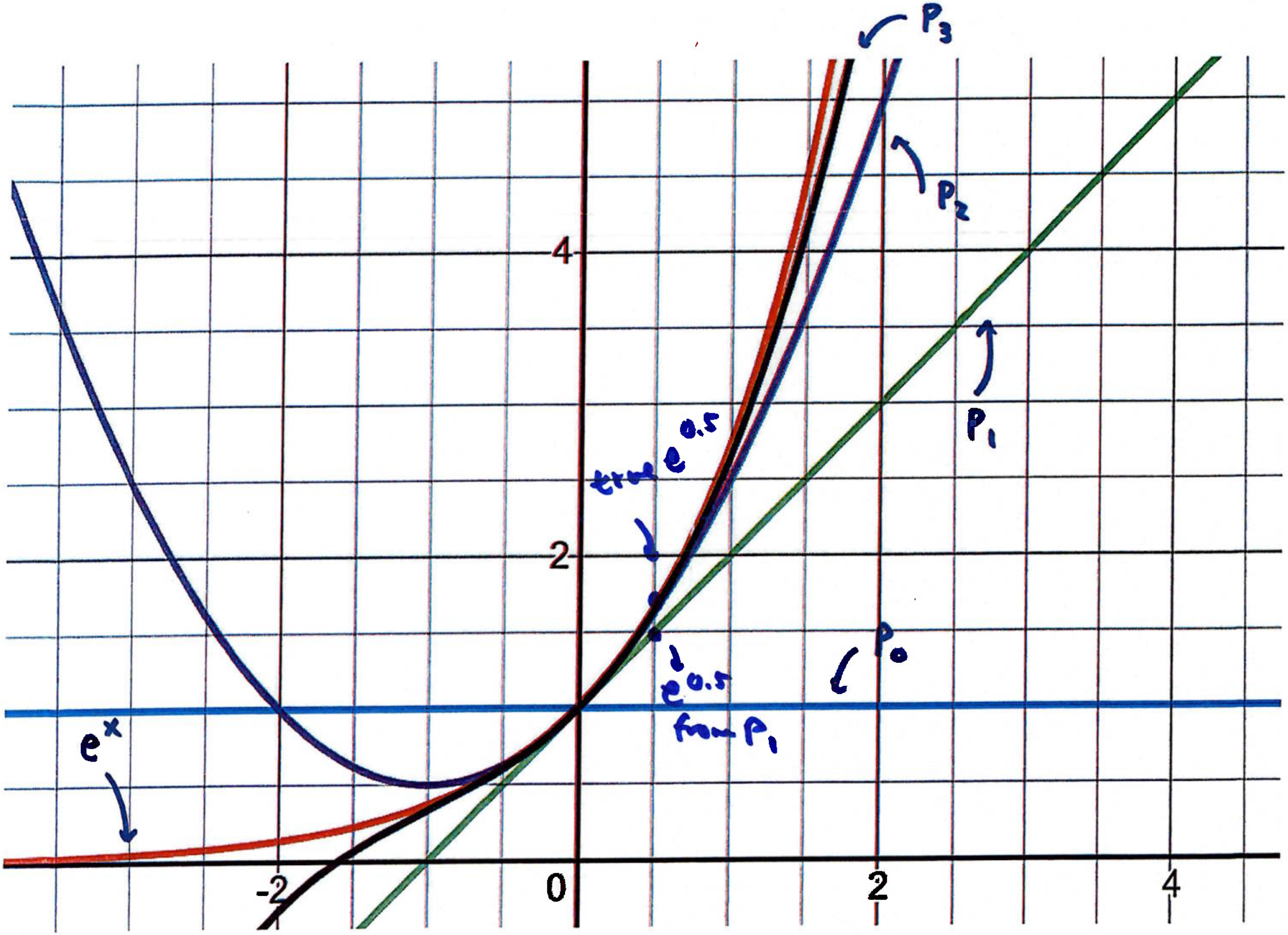
$$\left.\begin{aligned} P_0 &= 1 \\ P_1 &= 1+x \quad \text{(linear approx)} \\ P_2 &= 1+x+\frac{x^2}{2} \\ P_3 &= 1+x+\frac{x^2}{2}+\frac{x^3}{6} \end{aligned}\right\} \approx e^x \text{ near } x=0$$

one use of this : approx. $e^{0.5}$

w/o calculator, $e^{0.5} = ?$

but $P_1(x) \approx e^x \approx 1+x$

so $e^{0.5} \approx 1+0.5 \approx 1.5$ (true value is $e^{0.5} = 1.6487$)



P_3
P_2
4
P_1
true $e^{0.5}$
2
P_0
$e^{0.5}$ from P_1
e^x
-2
0
2
4

example Find the 4th-order Taylor polynomial of $f(x) = \cos(2x)$

near $x = a = \frac{\pi}{8}$

(so we want a 4th order polynomial that behaves like

$\cos(2x)$ near $x = \pi/8$)

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad a = \pi/8$$

up to $k = 4$

$$f(x) = f\left(\frac{\pi}{8}\right) + f'\left(\frac{\pi}{8}\right)\left(x - \frac{\pi}{8}\right) + \frac{1}{2!} f''\left(\frac{\pi}{8}\right)\left(x - \frac{\pi}{8}\right)^2$$

$$+ \frac{1}{3!} f'''\left(\frac{\pi}{8}\right)\left(x - \frac{\pi}{8}\right)^3 + \frac{1}{4!} f^{(4)}\left(\frac{\pi}{8}\right)\left(x - \frac{\pi}{8}\right)^4$$

$$f(x) = \cos(2x) \rightarrow f\left(\frac{\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -2\sin(2x) \rightarrow f'\left(\frac{\pi}{8}\right) = -2\sin\left(\frac{\pi}{4}\right) = -\sqrt{2}$$

$$f''(x) = -4\cos(2x) \rightarrow f''\left(\frac{\pi}{8}\right) = -4\cos\left(\frac{\pi}{4}\right) = -2\sqrt{2}$$

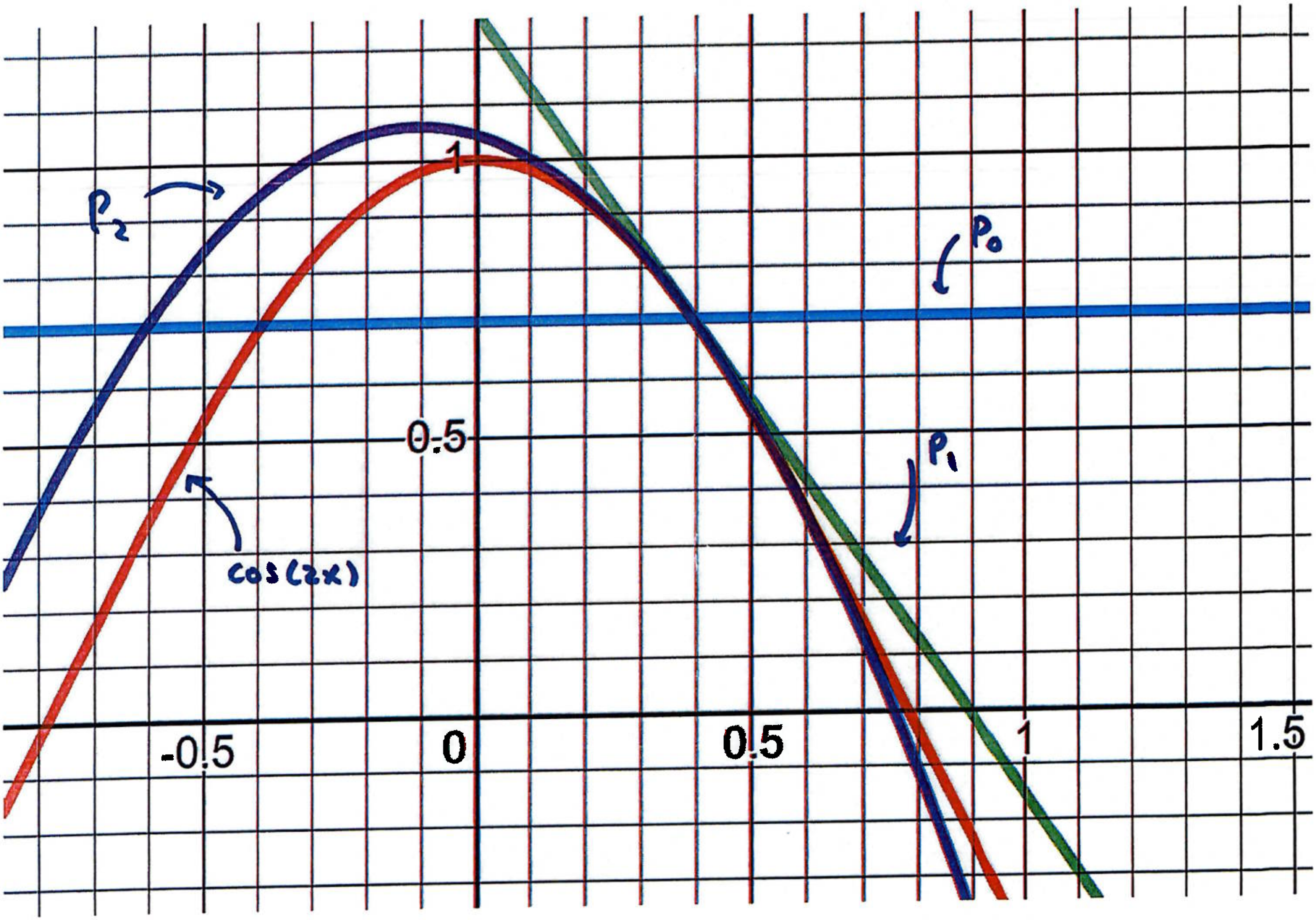
$$f'''(x) = 8\sin(2x) \rightarrow f'''\left(\frac{\pi}{8}\right) = 8 \cdot \frac{\sqrt{2}}{2} = 4\sqrt{2}$$

$$f^{(4)}(x) = 16\cos(2x) \rightarrow f''''\left(\frac{\pi}{8}\right) = 8\sqrt{2}$$

So, near $x = \frac{\pi}{8}$, $\cos(2x)$ behaves like

$$\underbrace{\frac{\sqrt{2}}{2}}_{f(\frac{\pi}{8})} - \underbrace{\sqrt{2}}_{f'(\frac{\pi}{8})}\left(x-\frac{\pi}{8}\right) - \frac{2\sqrt{2}}{2!}\left(x-\frac{\pi}{8}\right)^2 + \frac{4\sqrt{2}}{3!}\left(x-\frac{\pi}{8}\right)^3 + \frac{8\sqrt{2}}{4!}\left(x-\frac{\pi}{8}\right)^4$$

$$= \underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\frac{\sqrt{2}}{2}}_{P_0} - \sqrt{2}\left(x-\frac{\pi}{8}\right)}_{P_1} - \sqrt{2}\left(x-\frac{\pi}{8}\right)^2}_{P_2} + \frac{2\sqrt{2}}{3}\left(x-\frac{\pi}{8}\right)^3}_{P_3} + \frac{\sqrt{2}}{3}\left(x-\frac{\pi}{8}\right)^4}_{P_4}$$


1
0.5
-0.5
0
0.5
1
1.5
P2
P0
P1
cos(2x)

EXAM REVIEW
MON 11/6
6:30 - 8:30 pm
WTHR 200